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A255240: Archimedes's Construction of the Regular Heptagon

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In *Alten et al.* [1] and *Tropfke* [3] the construction of *Archimedes* for the regular heptagon, based on *neusis* (see Wikipedia [4]) is considered. A *neusis* construction is based on two curves K_1 and K_2, a point P and a distance s. The task is to find a straight line through P such that the two intersection points of the curves have distance s (compare [1], Abb. 2.4.6 on p. 86). The Abb. 2.4.7 on p. 86 of [1] and Abb. 78 on p. 430 of [3] have been redrawn here as *Figure 1*. In a first step one has to find the straight line $\overline{D,E}$ with slope $\tan\alpha$ in a square A, B, C, D (side length a) and one diagonal $\overline{A,C}$ such that the two areas F_1 of the triangle $\triangle(D,F,C)$ and F_2 of the right triangle $\triangle(G,B,E)$ become equal. The *neusis* data is here K_1: a straight line through A and B, K_2: a straight line containing the diagonal $\overline{A,C}$, $P = D$, L: a straight line through D, and $s = \overline{F,E}$ if $F_1 = F_2$. (Finding this s or α in the *neusis* construction will involve some approximation when comparing F_1 and F_2.)

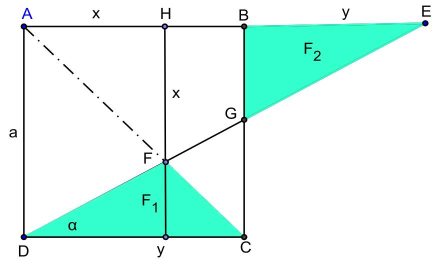

Figure 1: Finding α such that $F_1 = F_2$

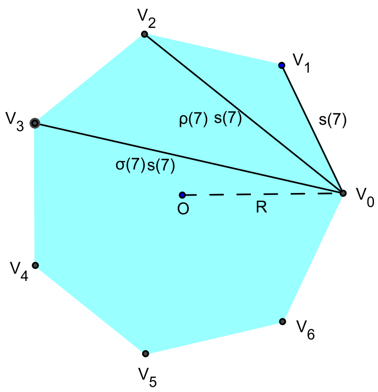

Figure 2: Heptagon

With $a = \overline{A,B} = \overline{A,D}$, $x = \overline{A,H}$ and $y = \overline{B,E}$ one finds for the scaled quantities $\hat{x} = \frac{x}{a}$ and $\hat{y} = \frac{y}{a}$ from $\frac{F_1}{a^2} = \frac{F_2}{a^2}$ and $\frac{1-\hat{x}}{\hat{x}} = \tan\alpha = \frac{\hat{x}}{1-\hat{x}+\hat{y}}$ the relations

$$\hat{y}^2 = \hat{x}, \quad \text{and} \quad (1-\hat{x})\,\hat{y} = 2\,\hat{x} - 1.$$

This leads to a cubic equation for $\hat{x}$, given in [1] as eq. 2.4.5 on p. 87 (after division by a^3),

$$\hat{x}^3 - 6\,\hat{x}^2 + 5\,\hat{x} - 1 = 0.$$

For $t := \tan\alpha = -1 + \frac{1}{\hat{x}}$ this translates to

$$t^3 - 2\,t^2 - t + 1 = 0,$$

or for $\rho = \frac{1}{t}$ to the minimal polynomial $C(7,n)$ of $\rho(7) := 2\cos(\frac{\pi}{7})$ (see [2], eq. (20), and *Tables 2* and *3*)

$$\rho^3 - \rho^2 - 2\,\rho + 1 = 0.$$

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Therefore, the three real zeros are $\rho_1 = \rho(7) = 1.801937736...$, $\rho_2 = 2\cos\left(\frac{3\pi}{7}\right) = -1 - \rho(7) + \rho(7)^2 \simeq .445$ and $\rho_3 = 2\cos\left(\frac{5\pi}{7}\right) = 2 - \rho(7)^2 \simeq -1.247$. For the decimal expansion of these zeros see A160389, A255241 and - A255249. The slope is $t = \tan\alpha = \frac{1}{\rho(7)} \simeq 0.55496$, corresponding to an angle $\alpha \simeq 29.028^o$. See A255240 for the decimal expansion of this slope. (The other solutions correspond to angles $\simeq 66.009^o$ and $\simeq -38.727^o$)

The scaled length $\hat{x}$ becomes, with $t = \dfrac{1}{\rho(7)} = 2 + \rho(7) - \rho(7)^2$, $\hat{x} = \dfrac{\rho(7)}{1 + \rho(7)} = \rho(7)\,(\rho(7)\,(2 - \rho(7))) = (\rho(7) - 1)^2 \simeq .6431$, due to the cubic equation for $\rho(7)$. Therefore,

$$\hat{y} = \sqrt{\hat{x}} = \rho(7) - 1 \simeq .8019, \quad \text{and} \quad \frac{\overline{A,E}}{a} = 1 + \hat{y} = \rho(7)\,.$$

The *neusis* distance s satisfies $\hat{s} := \dfrac{s}{a} = \dfrac{\overline{F,E}}{a} = \dfrac{\overline{F,G} + \overline{G,E}}{a}$. With $\dfrac{\overline{F,G}}{a} = \dfrac{1 - \hat{x}}{\cos\alpha} = (2 - \rho(7))\sqrt{1 + \rho(7)^2} \approx 0.40817$ and $\dfrac{\overline{G,E}}{a} = \dfrac{\hat{y}}{\cos\alpha} = \dfrac{\rho(7) - 1}{\rho(7)}\sqrt{1 + \rho(7)^2} = (-1 - \rho(7) + \rho(7)^2)\sqrt{1 + \rho(7)^2} \approx 0.91715$ we find $\hat{s} = (1 - 2\,\rho(7) + \rho(7)^2)\sqrt{1 + \rho(7)^2} \approx 1.32532\,.$

In the regular heptagon inscribed in a circle of radius R the length ratio of the small diagonal and the side $s(7) = R \cdot 2 \cdot \sin\left(\frac{\pi}{7}\right) = R\sqrt{4 - \rho(7)^2} \simeq 0.8678\,R$ is $\rho(7)$. The length ratio of the large diagonal and the side is $\sigma(7) = \rho(7)^2 - 1$ (see *e.g.*, [2] , *Table 1*, row $n = 7$). See *Figure 2*. For the decimal expansion of $\sigma(7)$ see A231187.

In a second step of the *neusis* construction based on $\overline{A,E}$ with the points H and B from above one finds (by ruler and a pair of compasses) the side length of a regular heptagon to be y as shown in *Figure 3*, adapted from [1], Abb. 2.4.9 on p. 87. This is shown as follows.

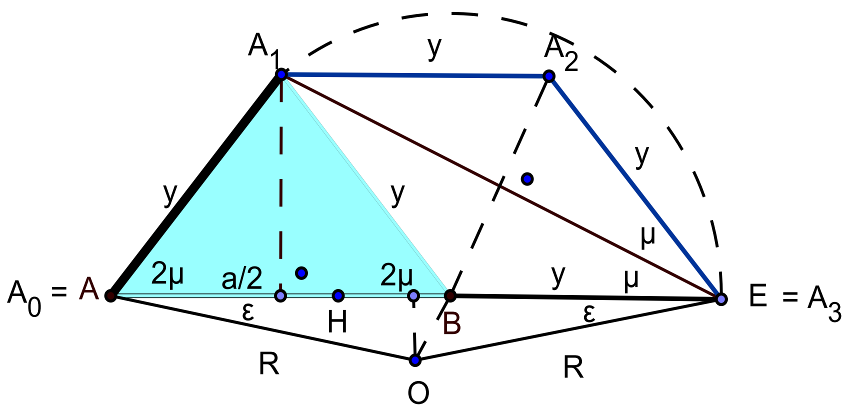

Figure 3: Finding the angle $\mu = \frac{\pi}{7}$ and $\varepsilon = \frac{\mu}{2}$

Point A_1 is the intersection point of the dashed vertical line originating at the midpoint of $\overline{A,B} = a$ and the dashed circle segment around B with radius $y = \overline{B,E}$. The colored (shaded) isosceles triangle $\triangle(A, B, A_1)$ has then angle $2\mu = \frac{2\pi}{7}$, because $\cos(2\,\mu) = \dfrac{a}{2\,y}$. *I.e.*, from above, using in the first

equation the cubic for $\rho(7)$, $-2 + \rho(7)^2 = \frac{1}{\rho(7) - 1} = \frac{1}{\hat{y}} = 2\cos(2\mu) = -2 + (2\cos\mu)^2$, hence $2\cos\mu = \rho(7) = 2\,cos\left(\frac{\pi}{7}\right)$, or here $\mu = \frac{\pi}{7}$.
The point A_2 is found by constructing the rhombus A_1, B, E, A_2. Because the angles $\angle(A, A_1, A_2)$ and $\angle(A_1, A_2, E)$ are both $\pi - 2\mu = \frac{5\pi}{7}$, part of the regular heptagon $A_0 = A, A_1, A_2, E = A_3$ with side length y has thus been constructed.

The ratio of the radius R of the circumscribing circle of the regular heptagon and the length a (of the original square of *Figure 1*) is found from $y = a\,\hat{y} = s(7) = R\sqrt{4 - \rho(7)^2}$.

$$\hat{R} := \frac{R}{a} = \frac{\hat{y}}{\sqrt{4 - \rho(7)^2}} = \frac{\sqrt{7}\,(\rho(7) - 1)}{-6 + \rho(7) + 2\,\rho(7)^2} = \frac{1}{\sqrt{7}}\left(1 - \rho(7) + \rho(7)^2\right) \simeq .9241\,.$$

We have used that $(-6 + \rho(7) + \rho(7)^2)^2 = 7\,(4 - \rho(7)^2)$ due to the cubic equation for $\rho(7)$ and similarly $\frac{1}{-6 + \rho(7) + \rho(7)^2} = \frac{1}{7}\,(-2 + \rho(7) + \rho(7)^2)$ and $(-1 + \rho(7))\,(-2 + \rho(7) + \rho(7)^2) = 1 - \rho(7) + \rho(7)^2$.
In *Figure 3* it is shown how to find the center O of the circumscribing circle. It is the intersection point of the straight line through $\overline{A_2, B}$ and the vertical through the midpoint of $\overline{A, E}$. Because the angle $\angle(A, O, E) = 3\,\frac{2\pi}{7}$ the angle $\varepsilon = \frac{\pi}{2} - \frac{3\,\pi}{7} = \frac{\pi}{14} = \frac{\mu}{2}$.
Knowing the center O and the radius R of the circumscribing circle it is now clear how to find the other three vertices A_4, A_5 and A_6 of the regular heptagon, with a pair of compasses alone, and the missing sides are drawn with a ruler. This then finishes *Archimedes*'s *neusis* construction of the regular heptagon.

References

[1] H.-W. Alten et al., *4000 Jahre Algebra*, 2. Auflage, Springer, 2014.

[2] W. Lang, The field $Q(2\cos(\pi/n))$, its Galois group and length ratios in the regular n-gon, October 2012, http://arxiv.org/abs/1210.1018.

[3] J. Tropfke, *Geschichte der Elementarmathematik*, Band 1. Arithmetik und Algebra, 4. Auflage, Walter de Gruyter, Berlin, New York, 1980.

[4] Wikipedia, Neusis Construction, https://en.wikipedia.org/wiki/Neusis_construction .

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Concerned with OEIS sequences A160389, A231187, A255240, A255241, A255249.