A254671 and A061743

Robert Israel

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A254671 is "Numbers that can be represented as $xy + x + y$, where $x \geq y \geq 1$." As noted by David A. Corneth, k is in the sequence if and only if $k+1$ is an odd composite or an even number with more than 4 divisors.

A061743 is "Numbers k such that $k!$ is divisible by $(k+1)^2$."

R. J. Mathar commented that A254671 was "apparently 8 and the elements of A061743". We prove this fact.

For any prime p, the p-adic order of $k!$ is $\nu_p(k!) = \sum_{j\geq 1} \lfloor k/p^j \rfloor$, and $k!$ is divisible by $(k+1)^2$ if and only if $\nu_p(k!) \geq \nu_p((k+1)^2) = 2\nu_p(k+1)$ for every prime p dividing $k+1$.

The first case to consider is when $k+1 = p^m$ is a prime power (with $m \geq 1$).

$$\nu_p(k!) = \sum_{j=1}^{m-1} \lfloor (p^m - 1)/p^j \rfloor = \sum_{j=1}^{m-1} (p^{m-j} - 1) = \frac{p^m - 1 - m(p-1)}{p-1}$$

so

$$\nu_p(k!) - 2\nu_p(k+1) = \frac{p^m - 1 - m(p-1)}{p-1} - 2m = \frac{p^m - 1 - 3m(p-1)}{p-1}$$

For $m = 1$, $p - 1 - 3(p-1) = 2 - 2p < 0$ for all primes p, thus no members k of A061743 have $k+1$ prime; the numbers with $k+1$ prime are also not in A254671.

For $m = 2$, $p^2 - 1 - 6(p-1) = p^2 - 6p + 5 = (p-5)(p-1) < 0$ only for primes $p = 2$ and 3, so $k = p^2 - 1$ is in A061743 for primes ≥ 5, and these are also in A254671, while for $p = 2$, 3 is in neither A061743 nor A254671, and for $p = 3$, $k = 8$ is the special case mentioned by Mathar that is in A254671 but not A061743.

For $m = 3$, $p^3 - 1 - 9(p-1) = p^3 - 9p + 8 < 0$ only for prime $p = 2$, corresponding to $k = 7$. Thus 7 is not in A061743, and $7 + 1 = 8$ is an even number with exactly 4 divisors so 7 is not in A254671, while for all other primes $p^3 - 1$ is in A061743 and (as p^3 is an odd composite) also in A254671.

For $m \geq 4$, $p^m - 1 - 3m(p-1) > 0$ for all $p \geq 2$. Since 2^m has more than 4 divisors, $k = p^m - 1$ is in both A061743 and A254671.

Now suppose $k+1$ is not a prime power. First we consider the case where $k+1$ is even, thus $k + 1 = 2^m b$ where $b > 1$ is odd. This has more than 4 divisors unless $m = 1$ and b is prime.

If $m = 1$, i.e. $k + 1 = 2b$, and b is an odd prime, $\nu_b((2b-1)!) = 1 < 2\nu_b(2b)$ so k is not in A061743 as well as not being in A254671.

If $k + 1 = 2^m b$ where $m > 1$ and $b \geq 3$ is odd,

$$\nu_2(k!) = \sum_{j=1}^{m} \lfloor k/2^j \rfloor \geq 2^{m-1}b - 1 \geq 2m = 2\nu_2(k+1)$$

because $b \geq 3$ and $2^m/m \geq 2 > 4/3$, while for any prime p dividing b, $\nu_p(k!) \geq (2^m - 1)\nu_p(b) > 2\nu_p(k+1)$. Thus in this case k is in both A061743 and A254671.

If b is composite, say $b = cd$ where $c \geq d \geq 3$ are odd, then $k+1$'s divisors 1, 2^m, c, $2^m d$, cd are all distinct and $\leq k$, so $k!$ is divisible by the product of these divisors, $2^{2m}c^2d^2 = (k+1)^2$, So here again k is in both A061743 and A254671. This completes the case where $k+1$ is even and not a prime power.

Finally, suppose $k+1$ is odd and not a prime power. Then we can write $k+1 = cd$ where $c, d \geq 3$ are coprime. As $k+1$ is an odd composite, k is in A254671. As c, $2c$, d, $2d$ are distinct and $\leq k$, $(k+1)!$ is divisible by their product $4c^2d^2$ and thus by $(k+1)^2$. Thus again k is in both A061743 and A254671.