## Pascal's triangle and recurrence relations for partial sums of $m$-th powers

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We build with Excel the table that calculates, for successive additions (left cell + top cell) the partial sums of powers of natural numbers:

| n | $\mathrm{n}^{\mathrm{m}}$ | $1^{\text {th }}$ sums | $2^{\text {th }}$ sums | $3^{\text {th }}$ sums |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| 2 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| 3 | $\cdots$ | $\cdots$ | d | $\cdots$ |
| 4 | $\cdots$ | b | e | $\cdots$ |
| 5 | a | c | f | $\cdots$ |
| 6 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| 7 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

We want to obtain the recurrence relation for the second sums, that is, a formula for calculating the n-th term in the column "2th sums" as a function of the previous terms.

The formula that we seek is obtained by analyzing the data in the table as follows:
$c=a+b$
$e=b+d$
$f=c+e=a+b+e=a+e-d+e$
$f=2 e-d+a$
Indicating with $\mathrm{a}(\mathrm{n}, \mathrm{m})$ the n -th term of the sequence, we therefore have:

2th sums: $\quad a_{(n, m)}=2 a_{(n-1, m)}-a_{(n-2, m)}+n^{m}$

Extending the previous scheme to the successive columns, one obtains:
3th sums: $\quad a_{(n, m)}=3 a_{(n-1, m)}-3 a_{(n-2, m)}+a_{(n-3, m)}+n^{m}$
4 th $_{\text {th }}$ sums: $\quad a_{(n, m)}=4 a_{(n-1, m)}-6 a_{(n-2, m)}+4 a_{(n-3, m)}-a_{(n-4, m)}+n^{m}$

From this point forward we continue (successfully) using Pascal's triangle, by alternating signs, with the following results:

5th sums: $a_{(n, m)}=5 a_{(n-1, m)}-10 a_{(n-2, m)}+10 a_{(n-3, m)}-5 a_{(n-4, m)}+a_{(n-5, m)}+n^{m}$
$6^{\text {th }}: a_{(n, m)}=6 a_{(n-1, m)}-15 a_{(n-2, m)}+20 a_{(n-3, m)}-15 a_{(n-4, m)}+6 a_{(n-5, m)}-a_{(n-6, m)}+n^{m}$
and so on ....
So, if we denote by $p$ the order number of the partial sums, its recurrence relation is obtained by the following general formula:
pth sums: $\quad a_{(n, m)}=\left[\sum_{k=0}^{p-1}(-1)^{k}\binom{p}{k+1} a_{(n-1-k, m)}\right]+n^{m}$

All recurrence relations are valid, by induction, for each ( $\mathrm{n}, \mathrm{m}$ ).

The (1) is a "non linear" recurrence relation between the sequence terms. Also exists a "linear" recurrence relation, obtainable from Pascal's triangle, having the following general formula:
$p^{\text {th } \text { sums: }} \quad a_{(n, m)}=\sum_{k=0}^{p+m-1}(-1)^{k}\binom{p+m}{k+1} a_{(n-1-k, m)}$

Example: for the "fourth partial sums of sixth powers" you get the linear relationship:

$$
\begin{aligned}
a_{(n)}= & 10 a_{(n-1)}-45 a_{(n-2)}+120 a_{(n-3)}-210 a_{(n-4)}+252 a_{(n-5)}-210 a_{(n-6)} \\
& +120_{(n-7)}-45 a_{(n-8)}+10 a_{(n-9)}-a_{(n-10)}
\end{aligned}
$$

The coefficient list of expressions obtained from (2) is shown in O E I S with the term "signature".

