Pascal’s triangle and recurrence relations
for partial sums of m-th powers

By Luciano Ancora

We build with Excel the table that calculates, for successive additions (left cell + top cell) the partial sums of powers of natural numbers:

<table>
<thead>
<tr>
<th>n</th>
<th>n^m</th>
<th>1^st sums</th>
<th>2^nd sums</th>
<th>3^rd sums</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>...</td>
<td>...</td>
<td>d</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>...</td>
<td>b</td>
<td>e</td>
<td>...</td>
</tr>
<tr>
<td>5</td>
<td>a</td>
<td>c</td>
<td>f</td>
<td>...</td>
</tr>
<tr>
<td>6</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>7</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

We want to obtain the recurrence relation for the second sums, that is, a formula for calculating the n-th term in the column "2^nd sums" as a function of the previous terms.

The formula that we seek is obtained by analyzing the data in the table as follows:

\[ c = a + b \]
\[ e = b + d \]
\[ f = c + e = a + b + e = a + e - d + e \]
\[ f = 2e - d + a \]

Indicating with \( a(n,m) \) the n-th term of the sequence, we therefore have:

2^nd sums:
\[
2n^2 \overset{(n,1)}{=} 2a_{(n-1,m)} - a_{(n-2,m)} + n^m
\]

Extending the previous scheme to the successive columns, one obtains:

3^rd sums:
\[
3a_{(n,m)} = 3a_{(n-1,m)} - 3a_{(n-2,m)} + a_{(n-3,m)} + n^m
\]

4^th sums:
\[
4a_{(n,m)} = 4a_{(n-1,m)} - 6a_{(n-2,m)} + 4a_{(n-3,m)} - a_{(n-4,m)} + n^m
\]
From this point forward we continue (successfully) using Pascal’s triangle, by alternating signs, with the following results:

5th sums:

\[ a_{(n,m)} = 5a_{(n-1,m)} - 10a_{(n-2,m)} + 10a_{(n-3,m)} - 5a_{(n-4,m)} + a_{(n-5,m)} + n^m \]

6th:

\[ a_{(n,m)} = 6a_{(n-1,m)} - 15a_{(n-2,m)} + 20a_{(n-3,m)} - 15a_{(n-4,m)} + 6a_{(n-5,m)} - a_{(n-6,m)} + n^m \]

and so on ….

So, if we denote by \( p \) the order number of the partial sums, its recurrence relation is obtained by the following general formula:

\[
p\text{th sums:} \quad a_{(n,m)} = \left[ \sum_{k=0}^{p-1} (-1)^k \binom{p}{k+1}a_{(n-1-k,m)} \right] + n^m \tag{1}
\]

All recurrence relations are valid, by induction, for each \((n, m)\).

The \((1)\) is a "non linear" recurrence relation between the sequence terms. Also exists a "linear" recurrence relation, obtainable from Pascal’s triangle, having the following general formula:

\[
p\text{th sums:} \quad a_{(n,m)} = \sum_{k=0}^{p+m-1} (-1)^k \binom{p+m}{k+1}a_{(n-1-k,m)} \tag{2}
\]

Example: for the "fourth partial sums of sixth powers" you get the linear relationship:

\[
a_{(n)} = 10a_{(n-1)} - 45a_{(n-2)} + 120a_{(n-3)} - 210a_{(n-4)} + 252a_{(n-5)} - 210a_{(n-6)} + 120a_{(n-7)} - 45a_{(n-8)} + 10a_{(n-9)} - a_{(n-10)}
\]

The coefficient list of expressions obtained from \((2)\) is shown in OEIS with the term "signature".