The 24-hedral Number

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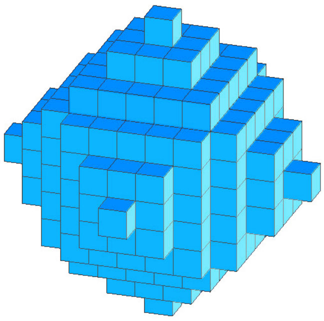
I built this figurate number by placing on a centered cube six square pyramids, as shown in the following figure:



The centered cube number in given by OEIS A016755:

$$CCub(n) = (2n+1)^3$$

The figurate number represented by the square pyramids is given by OEIS A000447:

$$P^{(4)}(n) = (1+n)(1+2n)(3+2n)/3$$

Therefore, for our 24-hedral number, we have:

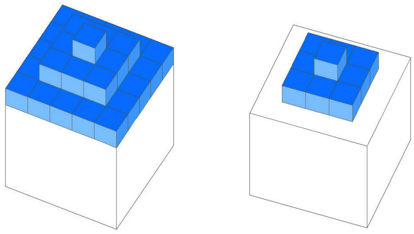
$$\begin{aligned} 24H(n) &= CCub(n) + 6P^{(4)}(n) \\ &= (2n+1)^3 + 6(1+n)(1+2n)(3+2n)/3 \\ &= (2n+1)(8n^2+14n+7) \qquad (1) \end{aligned}$$

The first few of these numbers are: 7, 87, 335, 847, 1719, 3047, 4927, ...

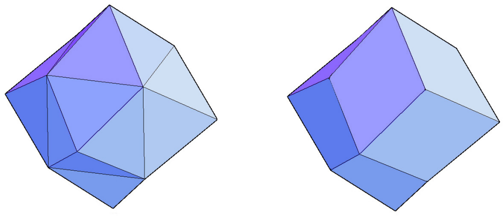
This figurate number is very close to the Hauy construction of the rhombic dodecahedron, given by OEIS A046142:

$$HauyRhoDod(n) = (2n-1)(8n^2-14n+7) \qquad (2)$$

The substantial difference between these figurate numbers lies in the fact that the HauyRhoDod construction uses $P^{(4)}(n-1)$ pyramids, as shown in the following figure (right part):



This difference, because of the different inclination of pyramids faces (in HauyRhoDod construction triangles are coplanar in pairs forming rhombs), produces two different constructions, a 24-hedron and a dodecahedron, respectively:



These two polyhedra are, for n large enough, almost coincident. In other words, as n tends to infinity, the 24-hedron tends to the dodecahedron, ie:

$$\lim_{n \to +\infty} 24H(n) = HauyRhoDod(n)$$

In fact, since the two third degree polynomials (1) and (2) have the same leading coefficient, their ratio tends to 1, as n tends to infinity.