Demonstration of formulas

We insert in Excel sequences of "Fourth partial sums of m-th powers", arranging them in a table as follows:

Fourth partial sums of m-th powers								
n	**m=0**	**m=1**	**m=2**	**m=3**	**m=4**	**m=5**	**m=6**	**m=7**
1	1	1	1	1	1	1	1	1
2	5	6	8	12	20	36	68	132
3	15	21	35	69	155	381	995	2709
4	35	56	112	272	760	2336	7672	26432
5	70	126	294	846	2814	10326	40614	168126
6	126	252	672	2232	8592	36552	166992	804552
7	210	462	1386	5214	22770	110022	571626	3136014
8	330	792	2640	11088	54120	292512	1701480	10459968
9	495	1287	4719	21879	117975	704847	4534959	30856839
10	715	2002	8008	40612	239668	1567852	11050468	82407052
11	1001	3003	13013	71643	459173	3263403	24997973	202678203
12	1365	4368	20384	121056	837200	6422208	53113424	465069696

Consider the recurrence relation:

$$a_{(n,m)} = 4a_{(n-1,m)} - 6a_{(n-2,m)} + 4a_{(n-3,m)} - a_{(n-4,m)} + n^m$$

which is valid, for each n, in every column of the table. We will use this relationship to derive formulas of the sequences that appear in each row of the table.

For $n = 2$ we have:

$$a_{(2,m)} = 4\times 1 - 0 + 0 - 0 + 2^m =$$

$$= \boxed{2^m + 4}$$

For $n = 3$:

$$a_{(3,m)} = 4(2^m + 4) - 6\times 1 + 0 - 0 + 3^m =$$

$$= \boxed{2^{m+2} + 3^m + 10}$$

For $n = 4$:

$$a_{(4,m)} = 4(2^{m+2} + 3^m + 10) - 6(2^m + 4) + 4 \times 1 - 0 + 4^m$$

$$= \boxed{2^{2m} + 5 \times 2^{m+1} + 4 \times 3^m + 20}$$

Continuing we get:

$$a_{(5,m)} = \boxed{5 \times 2^{m+2} + 2^{2m+2} + 10 \times 3^m + 5^m + 35}$$

$$a_{(6,m)} = \boxed{35 \times 2^m + 5 \times 2^{2m+1} + 20 \times 3^m + 4 \times 5^m + 6^m + 56}$$

$$a_{(7,m)} =$$

$$\boxed{7 \times 2^{m+3} + 5 \times 2^{2m+2} + 35 \times 3^m + 2^{m+2} \times 3^m + 2 \times 5^{m+1} + 7^m + 84}$$

and so on …..

This inductive process works indefinitely, generating polynomial expressions longer and longer, which in turn generate sequences with terms that magnify more and more rapidly.

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