Recurrence relations for partial sums of m-th powers

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We build with Excel the table that calculates, for successive additions (left cell + top cell) the partial sums of powers of natural numbers:

n	n^m	1^{th} sums	2^{th} sums	3^{th} sums
1	...	...	...	...
2	...	...	...	...
3	...	...	d	...
4	...	b	e	...
5	a	c	f	...
6	...	...	...	...
7	...	...	...	...

We want to obtain the recurrence relation for the second sums, that is, a formula for calculating the n-th term in the column "2th sums" as a function of the previous terms.

The formula that we seek is obtained by analyzing the data in the table as follows:

c = a + b
e = b + d
f = c + e = a + b + e = a + e – d + e

f = 2e – d + a

Indicating with a(n,m) the n-th term of the sequence, we therefore have:

2^{th} sums: $$a_{(n,m)} = 2a_{(n-1,m)} - a_{(n-2,m)} + n^m$$

Extending the previous scheme to the successive columns, one obtains:

3^{th} sums: $$a_{(n,m)} = 3a_{(n-1,m)} - 3a_{(n-2,m)} + a_{(n-3,m)} + n^m$$

4^{th} sums: $$a_{(n,m)} = 4a_{(n-1,m)} - 6a_{(n-2,m)} + 4a_{(n-3,m)} - a_{(n-4,m)} + n^m$$

From this point forward we continue (successfully) using Pascal's triangle, by alternating signs, with the following results:

5th sums: $$a_{(n,m)} = 5a_{(n-1,m)} - 10a_{(n-2,m)} + 10a_{(n-3,m)} - 5a_{(n-4,m)} + a_{(n-5,m)} + n^m$$

6th: $$a_{(n,m)} = 6a_{(n-1,m)} - 15a_{(n-2,m)} + 20a_{(n-3,m)} - 15a_{(n-4,m)} + 6a_{(n-5,m)} - a_{(n-6,m)} + n^m$$

and so on ….

All recurrence relations are valid, by *induction*, for each (n, m).