

Demonstration of formulas

We insert in Excel sequences of "Third partial sums of m-th powers", arranging them in a table as follows:

Third partial sums of m-th powers									
n	m=0	m=1	m=2	m=3	m=4	m=5	m=6	m=7	
1	1	1	1	1	1	1	1	1	1
2	4	5	7	11	19	35	67	131	
3	10	15	27	57	135	345	927	2577	
4	20	35	77	203	605	1955	6677	23723	
5	35	70	182	574	2054	7990	32942	141694	
6	56	126	378	1386	5778	26226	126378	636426	
7	84	210	714	2982	14178	73470	404634	2331462	
8	120	330	1254	5874	31350	182490	1129854	7323954	
9	165	495	2079	10791	63855	412335	2833479	20396871	
10	220	715	3289	18733	121693	863005	6515509	51550213	
11	286	1001	5005	31031	219505	1695551	13947505	120271151	
12	364	1365	7371	49413	378027	3158805	28115451	262391493	

Consider the [recurrence relation](#):

$$a_{(n,m)} = 3a_{(n-1,m)} - 3a_{(n-2,m)} + a_{(n-3,m)} + n^m$$

which is valid, for each n , in every column of the table. We will use this relationship to derive formulas of the sequences that appear in each row of the table.

For $n = 2$ we have:

$$\begin{aligned} a_{(2,m)} &= 3 \times 1 - 0 + 0 + 2^m = \\ &= 2^m + 3 \end{aligned}$$

For $n = 3$:

$$\begin{aligned} a_{(3,m)} &= 3(2^m + 3) - 3 \times 1 + 0 + 3^m = \\ &= 3 \times 2^m + 3^m + 6 \end{aligned}$$

For $n = 4$:

$$a_{(4,m)} = 3(3 \times 2^m + 3^m + 6) - 3(2^m + 3) + 1 + 4^m$$

$$= 2^{2m} + 3 \times 2^{m+1} + 3^{m+1} + 10$$

Continuing we get:

$$a_{(5,m)} = 3 \times 2^{2m} + 5 \times 2^{m+1} + 2 \times 3^{m+1} + 5^m + 15$$

$$a_{(6,m)} = 15 \times 2^m + 3 \times 2^{2m+1} + 10 \times 3^m + 3 \times 5^m + 6^m + 21$$

$$a_{(7,m)} = 21 \times 2^m + 5 \times 2^{2m+1} + 5 \times 3^{m+1} + 2^m \times 3^{m+1} + 6 \times 5^m + 7^m + 28$$

and so on

This inductive process works indefinitely, generating polynomial expressions longer and longer, which in turn generate sequences with terms that magnify more and more rapidly.

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