Maple-assisted proof of empirical formula for A251376

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There are $4^4 = 256$ possible rows, which I will number from 1 to 256. Let T be the 256×256 matrix such that $T_{ij} = 1$ if row i can be followed by row j, i.e. each 2×2 subblock formed by these rows has sum 5, 6 or 7, and $T_{ij} = 0$ otherwise. The following code produces the matrix T.

```
> Rows:= [seq(seq(seq(seq([a,b,c,d],a=0..3),b=0..3),c=0..3),d=0..3)
  ]:
  f:= proc(i,j) local S,k;
    S:= [seq(Rows[i][k]+Rows[i][k+1]+Rows[j][k]+Rows[j][k+1], k=1.
  .3)];
    if min(S)>=5 and max(S)<=7 then 1 else 0 fi
  end proc:
  T:= Matrix(256,256,f):
```

Then we should have $a(n) = e^T T^n e$ where e is the vector of all 1's. To confirm, we compute the first few entries. For future use, $U_j = T^j e$.

```
> U[0]:= Vector(256,1):
  for j from 1 to 52 do U[j]:= T . U[j-1] od:
  seq(U[0]^%T . U[j], j=1..13);
```

9124, 385704, 17309672, 796635224, 37078015004, 1734708707700, 81356563852940, 3819965043029148, 179459290594576812, 8433120201697489676, 396339147306043331956, 18628300842155979771596, 875574504616798793124900 **(1)**

Now the empirical formula is

```
> Emp:= a(n)=106*a(n-1)-3549*a(n-2)+21133*a(n-3)+1140728*a(n-4)
  -20018738*a(n-5)-49586791*a(n-6)+3346494662*a(n-7)-13723317144*a
  (n-8)-213778801241*a(n-9)+1639122932515*a(n-10)+5825524717372*a
  (n-11)-78208197832246*a(n-12)-28235333475144*a(n-13)
  +2023100001243086*a(n-14)-2267952242440452*a(n-15)
  -30935552592134037*a(n-16)+62859381807362994*a(n-17)
  +286893688473269262*a(n-18)-793984601053515793*a(n-19)
  -1606418029904184151*a(n-20)+5799447065827370945*a(n-21)
  +5226126316716961173*a(n-22)-26467885691077674454*a(n-23)
  -8305459703842203946*a(n-24)+78758139329479805889*a(n-25)
  -1884170423271857427*a(n-26)-156951883602457225266*a(n-27)
  +36268479632044743829*a(n-28)+212781948689491583267*a(n-29)
  -77859864043892779649*a(n-30)-197232271880364557634*a(n-31)
  +90966031388418389501*a(n-32)+124073759715081468649*a(n-33)
  -66907064688342441620*a(n-34)-51686855942406206349*a(n-35)
  +32101010065557719435*a(n-36)+13500405320664544906*a(n-37)
  -10001970660967276596*a(n-38)-1935476096398211816*a(n-39)
  +1957664262699224096*a(n-40)+82594595240482592*a(n-41)
  -225625338538807552*a(n-42)+13112933329308160*a(n-43)
  +13831026197124096*a(n-44)-1701096470583808*a(n-45)
  -401666050397184*a(n-46)+71704231286784*a(n-47)+3881417293824*a
  (n-48)-1173282471936*a(n-49)+17717723136*a(n-50)+584456016*a
  (n-51)-254803968*a(n-52):
```

This corresponds to saying $e^T T^n P(T)\ e = 0$ where P is the following polynomial.

```
> P:= t^52 - add(coeff(rhs(Emp),a(n-j))*t^(52-j),j=1..52);
```

$$
\begin{aligned}
P := {} & t^{52} - 106\,t^{51} + 3549\,t^{50} - 21133\,t^{49} - 1140728\,t^{48} + 20018738\,t^{47} + 49586791\,t^{46} \\
& - 3346494662\,t^{45} + 13723317144\,t^{44} + 213778801241\,t^{43} - 1639122932515\,t^{42} \\
& - 5825524717372\,t^{41} + 78208197832246\,t^{40} + 28235333475144\,t^{39} \\
& - 2023100001243086\,t^{38} + 2267952242440452\,t^{37} + 30935552592134037\,t^{36} \\
& - 62859381807362994\,t^{35} - 286893688473269262\,t^{34} + 793984601053515793\,t^{33} \\
& + 1606418029904184151\,t^{32} - 5799447065827370945\,t^{31} - 5226126316716961173\,t^{30} \\
& + 26467885691077674454\,t^{29} + 8305459703842203946\,t^{28} - 78758139329479805889\,t^{27} \\
& + 18841704232718574 27\,t^{26} + 156951883602457225266\,t^{25} \\
& - 362684796320447438 29\,t^{24} - 212781948689491583267\,t^{23} \\
& + 778598640438927796 49\,t^{22} + 197232271880364557634\,t^{21} \\
& - 909660313884183895 01\,t^{20} - 124073759715081468649\,t^{19} \\
& + 669070646883424416 20\,t^{18} + 516868559424062063 49\,t^{17} \\
& - 321010100655577194 35\,t^{16} - 135004053206645449 06\,t^{15} \\
& + 100019706609672765 96\,t^{14} + 193547609639821181 6\,t^{13} - 195766426269922409 6\,t^{12} \\
& - 8259459524048259 2\,t^{11} + 2256253385388075 52\,t^{10} - 1311293332930816 0\,t^{9} \\
& - 1383102619712409 6\,t^{8} + 1701096470583808\,t^{7} + 401666050397184\,t^{6} \\
& - 71704231286784\,t^{5} - 3881417293824\,t^{4} + 1173282471936\,t^{3} - 17717723136\,t^{2} \\
& - 5844566016\,t + 254803968
\end{aligned}
\qquad (2)
$$

We compute $v = P(T)\ e$ using the previously computed values U_j.

```
> v:= add(coeff(P,t,j)*U[j],j=0..52):
```

The proof is completed by verifying that $v = 0$.

```
> v^%T . v;
```

$$0 \qquad (3)$$