Ornstein-Uhlenbeck Process

Steven Finch

May 15, 2004

We first define several words. A stochastic process $\{Y_t : t \geq 0\}$ is

- **stationary** if, for all $t_1 < t_2 < \ldots < t_n$ and $h > 0$, the random n-vectors $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ and $(Y_{t_1+h}, Y_{t_2+h}, \ldots, Y_{t_n+h})$ are identically distributed; that is, time shifts leave joint probabilities unchanged

- **Gaussian** if, for all $t_1 < t_2 < \ldots < t_n$, the n-vector $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ is multivariate normally distributed

- **Markovian** if, for all $t_1 < t_2 < \ldots < t_n$, $\mathrm{P}(Y_{t_n} \leq y \,|\, Y_{t_1}, Y_{t_2}, \ldots, Y_{t_{n-1}}) = \mathrm{P}(Y_{t_n} \leq y \,|\, Y_{t_{n-1}})$; that is, the future is determined only by the present and not the past.

Also, a process $\{Y_t : t \geq 0\}$ is said to have **independent increments** if, for all $t_0 < t_1 < \ldots < t_n$, the n random variables $Y_{t_1} - Y_{t_0}$, $Y_{t_2} - Y_{t_1}$, ..., $Y_{t_n} - Y_{t_{n-1}}$ are independent. This condition implies that $\{Y_t : t \geq 0\}$ is Markovian, but not conversely. The increments are further said to be **stationary** if, for any $t > s$ and $h > 0$, the distribution of $Y_{t+h} - Y_{s+h}$ is the same as the distribution of $Y_t - Y_s$. This additional provision is needed for the following definition.

A stochastic process $\{W_t : t \geq 0\}$ is a **Wiener-Lévy process** or **Brownian motion** if it has stationary independent increments, if W_t is normally distributed and $\mathrm{E}(W_t) = 0$ for each $t > 0$, and if $W_0 = 0$. It follows immediately that $\{W_t : t > 0\}$ is Gaussian and that $\mathrm{Cov}(W_s, W_t) = \theta^2 \min\{s, t\}$, where the variance parameter θ^2 is a positive constant. For concreteness' sake, we henceforth assume that $\theta = 1$. Almost all sample paths of Brownian motion are everywhere continuous but nowhere differentiable.

One technical stipulation is required for the following. A stochastic process $\{Y_t : t \geq 0\}$ is **continuous in probability** if, for all $u \in \mathbb{R}^+$ and $\varepsilon > 0$, $\mathrm{P}\left(|Y_v - Y_u| \geq \varepsilon\right) \to 0$ as $v \to u$. This holds if $\mathrm{Cov}(Y_s, Y_t)$ is continuous over $\mathbb{R}^+ \times \mathbb{R}^+$. Note that this is a statement about distributions, not sample paths.

Having dispensed with preliminaries, we turn to the central topic. A stochastic process $\{X_t : t \geq 0\}$ is an **Ornstein-Uhlenbeck process** or a **Gauss-Markov**

[0]Copyright © 2004 by Steven R. Finch. All rights reserved.

process if it is stationary, Gaussian, Markovian, and continuous in probability [1, 2]. A fundamental theorem, due to Doob [3, 4, 5], ensures that $\{X_t : t \geq 0\}$ necessarily satisfies the following linear stochastic differential equation:

$$dX_t = -\rho(X_t - \mu)dt + \sigma\, dW_t$$

where $\{W_t : t \geq 0\}$ is Brownian motion with unit variance parameter and μ, ρ, σ are constants. We have moments

$$\mathrm{E}(X_t) = \mu, \qquad \mathrm{Cov}(X_s, X_t) = \frac{\sigma^2}{2\rho} e^{-\rho|s-t|}$$

in the unconditional (strictly stationary) case and

$$\mathrm{E}(X_t \,|\, X_0 = c) = \mu + (c - \mu)e^{-\rho t}$$

$$\mathrm{Cov}(X_s, X_t \,|\, X_0 = c) = \frac{\sigma^2}{2\rho}\left(e^{-\rho|s-t|} - e^{-\rho(s+t)}\right)$$

in the conditional (asymptotically stationary) case, where X_0 is initially constant. The latter case encompasses Brownian motion when $\mu = c = 0$, $\sigma = 1$ and $\rho \to 0^+$. The former case encompasses idealized **white noise** $\{dW_t/dt : t \geq 0\}$ when $\mu = 0$, $\sigma = \rho$ and $\rho \to \infty$.

Before proceeding, we note the following simple algorithm for generating a sample path of the Ornstein-Uhlenbeck process (also known as **colored noise**) over the time interval $[0, T]$. Let N be a large integer and let z_0, z_1, ..., z_N be independent random numbers generated from a normal distribution with mean 0 and variance $\sigma^2/(2\rho)$. Define $x_0 = \mu + z_0$ for the unconditional case and $x_0 = c$ for the conditional case. Then define recursively

$$x_n = \mu + \kappa_N(x_{n-1} - \mu) + \sqrt{1 - \kappa_N^2}\, z_n$$

for $1 \leq n \leq N$, where $\kappa_N = \exp(-\rho T/N)$. The sequence x_0, x_1, ..., x_N is called a first-order autoregressive sequence (a discrete analog of the OU process) with lag-one correlation coefficient κ_N. Finally, interpolate linearly the values $X(nT/N) = x_n$ for $0 \leq n \leq N$ to obtain the desired path [6, 7, 8]. More sophisticated simulation methods are found in [9, 10, 11].

For concreteness' sake, we henceforth assume that $\mu = 0$, $\rho = 1$ and $\sigma^2 = 2$. (Some authors take $\sigma^2 = 1$ instead; the decision becomes apparent in any paper by seeing whether $\mathrm{Cov}(X_s, X_t)$ is $e^{-|s-t|}$ or $e^{-|s-t|}/2$.) The conditional probability

$$\mathrm{P}(X_t \leq x \,|\, X_0 = c) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \int_{-\infty}^{x} \exp\left(-\frac{(\xi - ce^{-t})^2}{2(1 - e^{-2t})}\right) d\xi$$

tends to the standard normal distribution, of course, as $t \to \infty$ (meaning that transients die out with time and don't affect long-term behavior). Likewise, $\mathrm{P}(X_s \leq x$ and $X_t \leq y \mid X_0 = c)$ can be evaluated. One might believe that the solution of any problem involving the OU process would be similarly straightforward; the following sections serve, however, to eliminate such ideas [12, 13].

0.1. First-Passage Times. For $a \in \mathbb{R}$, we wish to find the length of time required for an OU process to cross the level $x = a$, given that it started at $x = c$. Define the **first-passage time** or **hitting time** $T_{a,c}$ by $T_{a,c} = \inf\{t \geq 0 : X_t = a \mid X_0 = c\}$. The random variable $T_{a,c}$ is 0 if and only if $a = c$. Let $f_{a,c}(t)$ denote the density function of $T_{a,c}$. In the special case when $a = 0$, it is known that [2, 12, 14, 15]

$$f_{0,c}(t) = \sqrt{\frac{2}{\pi}} \frac{|c| e^{-t}}{(1 - e^{-2t})^{3/2}} \exp\left(-\frac{c^2 e^{-2t}}{2(1 - e^{-2t})}\right)$$

but for $a \neq 0$, the formulas for $f_{a,c}(t)$ are more complicated (as we shall soon see). For $a > 0$ and $c > 0$, Thomas [16] and Ricciardi & Sato [17, 18] demonstrated that

$$\mathrm{E}(T_{a,0}) = \sqrt{\frac{\pi}{2}} \int\limits_0^a \left(1 + \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right)\right) \exp\left(\frac{t^2}{2}\right) dt = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\left(\sqrt{2}a\right)^k}{k!} \Gamma\left(\frac{k}{2}\right),$$

$$\mathrm{E}(T_{0,c}) = \sqrt{\frac{\pi}{2}} \int\limits_{-c}^0 \left(1 + \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right)\right) \exp\left(\frac{t^2}{2}\right) dt = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left(\sqrt{2}c\right)^k}{k!} \Gamma\left(\frac{k}{2}\right)$$

and, for example,

$$\begin{aligned} &\mathrm{E}(T_{1,0}) = 2.0934066496..., && \mathrm{E}(T_{0,1}) = 0.9019080126..., \\ &\mathrm{E}(T_{2,0}) = 10.4284093979..., && \mathrm{E}(T_{0,2}) = 1.4252045655.... \end{aligned}$$

The asymmetry in going from 0 to x, versus going from x to 0, is unsurprising: The process has mean 0, hence it tends to arrive at 0 more often than it departs from 0. For $a > 0$ and $c > 0$, we have [17, 18]

$$\begin{aligned} \mathrm{Var}(T_{a,0}) &= \sqrt{2\pi} \int\limits_0^a \int\limits_{-\infty}^t \int\limits_s^a \left(1 + \operatorname{erf}\left(\frac{r}{\sqrt{2}}\right)\right) \exp\left(\frac{r^2 + t^2 - s^2}{2}\right) dr\, ds\, dt - \mathrm{E}(T_{a,0})^2 \\ &= \mathrm{E}(T_{a,0})^2 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{\left(\sqrt{2}a\right)^k}{k!} \Gamma\left(\frac{k}{2}\right) \Psi\left(\frac{k}{2}\right), \end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T_{0,c}) &= \sqrt{2\pi}\int_{-c}^{0}\int_{-\infty}^{t}\int_{s}^{0}\left(1+\mathrm{erf}\left(\frac{r}{\sqrt{2}}\right)\right)\exp\left(\frac{r^2+t^2-s^2}{2}\right)dr\,ds\,dt-\mathrm{E}(T_{0,c})^2 \\ &= \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{\left(\sqrt{2}c\right)^k}{k!}\Gamma\left(\frac{k}{2}\right)\Psi\left(\frac{k}{2}\right)-\mathrm{E}(T_{0,c})^2\end{aligned}$$

where $\Psi(x) = \psi(x) - \psi(1)$ and $\psi(x)$ is the digamma function [19]. In particular, $\Psi(1) = 0$ and

$$\Psi(x) = \begin{cases} \displaystyle\sum_{j=1}^{x-1}\frac{1}{j} & \text{if } x \text{ is an integer} > 1 \\ \displaystyle -2\ln(2) + 2\sum_{j=1}^{x-1/2}\frac{1}{2j-1} & \text{if } x \text{ is a half-integer} > 0. \end{cases}$$

For example,

$$\begin{array}{ll} \mathrm{Var}(T_{1,0}) = 5.8420278024..., & \mathrm{Var}(T_{0,1}) = 0.8510837032..., \\ \mathrm{Var}(T_{2,0}) = 105.2752035488..., & \mathrm{Var}(T_{0,2}) = 1.0669454393.... \end{array}$$

To compute $f_{a,c}(t)$ exactly for arbitrary a and c, we would need to invert the following (Laplace transform) identity due to Darling & Siegert [20, 21, 22, 23]:

$$\mathrm{E}(e^{-\lambda T_{a,c}}) = \int_0^{\infty} f_{a,c}(t)e^{-\lambda t}dt = \begin{cases} \dfrac{D_{-\lambda}(-c)}{D_{-\lambda}(-a)}\exp\left(\dfrac{c^2-a^2}{4}\right) & \text{if } c < a \\ \dfrac{D_{-\lambda}(c)}{D_{-\lambda}(a)}\exp\left(\dfrac{c^2-a^2}{4}\right) & \text{if } c > a \end{cases}$$

where $D_\nu(x)$ is the **parabolic cylinder function** or **Weber function** [24]:

$$D_\nu(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\exp\left(\dfrac{x^2}{4}\right)\displaystyle\int_0^{\infty} t^{\nu}\exp\left(-\frac{t^2}{2}\right)\cos\left(xt-\frac{\nu\pi}{2}\right)dt & \text{if } \nu > -1 \\ \dfrac{1}{\Gamma(-\nu)}\exp\left(-\dfrac{x^2}{4}\right)\displaystyle\int_0^{\infty} t^{-\nu-1}\exp\left(-\frac{t^2}{2}-xt\right)dt & \text{if } \nu < 0. \end{cases}$$

The two branches of this formula agree for $-1 < \nu < 0$. A differential equation

$$\frac{d^2y}{dx^2} - \left(\frac{x^2}{4}-\nu-\frac{1}{2}\right)y(x) = 0$$

is satisfied by $D_\nu(x)$ and, if ν is not an integer, independently by $D_{-\nu}(x)$. A series representation in terms of confluent hypergeometric functions [0.4] is also useful. Unfortunately a closed-form expression for the inverse Laplace transform seems not to be possible; only a numerical approach is feasible at present. Keilson & Ross [25] tabulated the distribution of $T_{a,c}$ for a number of values a and c. For example, the median time for an OU process X_t to reach $a = 1$, given that $X_0 = c = 0$, is 1.1892.... This corresponds to the 50$^{\text{th}}$ percentile of the distribution of $T_{1,0}$. The median of $T_{2,0}$, by contrast, is 7.2521....

We turn to a more complicated problem involving two (absorbing) boundaries rather than just one. Given $a < c < b$, what is the length of time required for the process to escape the interval (a, b), given that it started at $x = c$? Define $T_{a,b,c} = \inf\{t \geq 0 : X_t = a \text{ or } X_t = b \,|\, X_0 = c\}$ and let $f_{a,b,c}(t)$ denote the density function of $T_{a,b,c}$. Efforts have focused on the scenario in which $-a = b > 0$. The Laplace transform of $f_{-b,b,c}(t)$ satisfies [22]

$$\mathrm{E}(e^{-\lambda T_{-b,b,c}}) = \frac{D_{-\lambda}(c) + D_{-\lambda}(-c)}{D_{-\lambda}(b) + D_{-\lambda}(-b)} \exp\left(\frac{c^2 - b^2}{4}\right)$$

assuming $-b < c < b$. From another table in [25], the median of $T_{-1,1,0}$ is found to be 0.4449.... The reason that this is less than 1.1892... is clear: Each direction of travel leads to a potential crossing. The median of $T_{-2,2,0}$ is 3.2439....

Keilson & Ross' approach to evaluating such probabilities was based on finding zeroes and residues in the complex plane of the parabolic cylinder functions. Alternative approaches for numerically computing $f_{a,c}(t)$ and $f_{-b,b,c}(t)$ include [26, 27, 28, 29]. We report on some related asymptotics in [0.4].

There is an obvious connection between first-passage times and extreme values of a process (in the conditional case). We simply summarize:

$$\left.\begin{array}{ll} \mathrm{P}\left(\max\limits_{0 \leq t \leq T} X_t \leq a \,\middle|\, X_0 = c\right) & \text{if } c < a \\ \mathrm{P}\left(\min\limits_{0 \leq t \leq T} X_t \geq a \,\middle|\, X_0 = c\right) & \text{if } c > a \end{array}\right\} = \mathrm{P}(T_{a,c} > T) = 1 - F_{a,c}(T)$$

and, if $a < c < b$,

$$\mathrm{P}\left(a \leq \min_{0 \leq t \leq T} X_t \leq \max_{0 \leq t \leq T} X_t \leq b \,\middle|\, X_0 = c\right) = \mathrm{P}(T_{a,b,c} > T) = 1 - F_{a,b,c}(T)$$

where $F_{a,c}(t)$, $F_{a,b,c}(t)$ are the cumulative distribution functions of $T_{a,c}$, $T_{a,b,c}$. In the special case when $-a = b > 0$, the latter formula becomes a statement about

$\max_{0\le t\le T}|X_t|$, given $X_0=c$. Also, the **range** of the process satisfies [22]

$$\mathrm{P}\left(\max_{0\le t\le T}X_t-\min_{0\le t\le T}X_t\le r\,\middle|\,X_0=c\right)=\int\limits_0^r\int\limits_{c-q}^c\left.\frac{\partial^2}{\partial a\,\partial b}F_{a,b,c}(T)\right|_{b=a+q}da\,dq$$

but no one apparently has calculated this probability.

0.2. Historical Maximums. If the condition $X_0=c$ is discarded, what then can be said about $\max_{0\le t\le T}X_t$ or $\max_{0\le t\le T}|X_t|$? We focus solely on the former expression and write $M_T=\max_{0\le t\le T}X_t$. It can be shown that [30, 31, 32]

$$\mathrm{P}(M_T\le 0)=\frac{1}{\pi}\arcsin\left(e^{-T}\right)$$

which is a beautiful (but isolated) result. More generally [32],

$$\int\limits_0^\infty\mathrm{P}(M_t\le y)e^{-\lambda t}dt=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^y\frac{1}{\lambda}\left(1-\frac{D_{-\lambda}(-x)}{D_{-\lambda}(-y)}\exp\left(\frac{x^2-y^2}{4}\right)\right)\exp\left(-\frac{x^2}{2}\right)dx$$

for arbitrary y, or

$$\int\limits_0^\infty g_t(y)e^{-\lambda t}dt=\frac{1}{\sqrt{2\pi}}\frac{D_{-\lambda-1}(-y)^2}{D_{-\lambda}(-y)^2}\exp\left(\frac{-y^2}{2}\right)$$

where $g_t(y)$ is the density function of M_t. For example, the median value of M_1 is 1.0393... and the median value of M_{10} is 2.2202.... It can be inferred from [0.3] that the median of M_T is asymptotically $\sqrt{2\ln(T)}$ as $T\to\infty$.

An alternative approach for numerically computing $\mathrm{P}(M_t\le y)$ via the Mellin transform is due to DeLong [33, 34, 35]. We hope to report on this later. An interesting application to computer science, involving the maximum size reached by a dynamic data structure over a long span of time, is described in [36].

0.3. Pickands' Constants. Assume that $\{Y_t:t\ge 0\}$ is a stationary Gaussian process with zero mean, unit variance and covariance function of the form

$$r\left(|s-t|\right)=\mathrm{Cov}(Y_s,Y_t)=1-C\,|s-t|^\alpha+o\left(|s-t|^\alpha\right)$$

as $|s-t|\to 0$, where $0<\alpha\le 2$ and $C>0$ are constants. Assume further that $r(\tau)\ln(\tau)\to 0$ as $\tau\to\infty$. Pickands [37, 38, 39, 40, 41] demonstrated that $M_T=\max_{0\le t\le T}Y_t$ has the Gumbel limiting distribution [42]

$$\lim_{T\to\infty}\mathrm{P}\left(\sqrt{2\ln(T)}\,(M_T-k_T)\le x\right)=\exp(-e^{-x})$$

where

$$k_T = \sqrt{2\ln(T)} + \frac{1}{\sqrt{2\ln(T)}}\left\{\frac{2-\alpha}{2\alpha}\ln(\ln(T)) + \ln\left((2\pi)^{-\frac{1}{2}}2^{\frac{2-\alpha}{2\alpha}}C^{\frac{1}{\alpha}}H_\alpha\right)\right\}$$

and H_α is a positive constant independent of C. It is known that $H_1 = 1$ (corresponding to the OU process) and $H_2 = 1/\sqrt{\pi}$. No other exact values for H_α are known. An alternative characterization of H_α is

$$H_\alpha = \lim_{T\to\infty}\int_0^\infty \mathrm{P}(\tilde{M}_T > y)e^y dy$$

where $\{\tilde{Y}_t : t \geq 0\}$ is a nonstationary Gaussian process with

$$\mathrm{E}(\tilde{Y}_t) = -\left|t\right|^\alpha, \qquad \mathrm{Cov}(\tilde{Y}_s, \tilde{Y}_t) = \left|s\right|^\alpha + \left|t\right|^\alpha - \left|s-t\right|^\alpha$$

but this does not seem to help. Shao [43] and Debicki, Michna & Rolski [44] gave bounds on H_α; for example,

$$0.009 \leq H_{1/2} \leq 715.94, \qquad 0.208 \leq H_{3/2} \leq 3.04.$$

A conjecture that $H_\alpha = 1/\Gamma(1/\alpha)$ remains unproved. There is also a connection with the Gaussian correlation conjecture and with estimating small ball probabilities [45], topics which we hope to address later.

0.4. Upper Tail Asymptotics. We revisit the single-boundary first-passage time distribution and ask about the limiting value

$$\lambda(a) = \lim_{t\to\infty}\frac{1}{t}\ln\left\{\mathrm{P}\left(T_{a,0} > t\right)\right\}$$

as a function of $a > 0$. In words, what can be said about the upper tail of the distribution of the first hitting time $T_{a,0}$ for an OU process X_t across the level $x = a$, given that $X_0 = 0$? Mandl [46, 47] and Beekman [48] demonstrated that $-1 < \lambda(a) < 0$ and that $\lambda(a)$ is the zero of $D_{-\lambda}(-a)$ closest to 0. Sample values include [17, 49, 50]

$$\lim_{a\to0^+}\lambda(a) = -1, \qquad \lim_{a\to\infty}\lambda(a)\cdot\frac{\exp(a^2/2)}{a} = \frac{-1}{\sqrt{2\pi}},$$

$$\lambda(0.7649508673...) = -\tfrac{1}{2},$$

$$\lambda(1) = -0.3882382947... = 2(-0.1941191473...),$$

$$\lambda(2) = -0.0972745958... = 2(-0.0486372979...).$$

For the symmetric double-boundary first-passage time distribution, we examine

$$\lambda(-b,b)=\lim_{t\to\infty}\frac{1}{t}\ln\left\{\mathrm{P}\left(T_{-b,b,0}>t\right)\right\}$$

as a function of $b>0$. Breiman [51] proved that $-\infty<\lambda(-b,b)<0$ and that $\lambda(-b,b)$ is the zero of $\Phi(\lambda/2,1/2,b^2/2)$ closest to 0, where

$$\Phi(u,v,w)=1+\sum_{k=1}^{\infty}\frac{u(u+1)(u+2)\cdots(u+k-1)}{v(v+1)(v+2)\cdots(v+k-1)}\frac{w^k}{k!}$$

is the **confluent hypergeometric function of the first kind**. For simplicity, define $\mu(b)=\lambda(-b,b)$. Sample values include [50, 51, 52]

$$\lim_{b\to0^+}\mu(b)=-\infty,\qquad\lim_{b\to\infty}\mu(b)\cdot\frac{\exp(b^2/2)}{b}=\frac{-1}{\sqrt{2\pi}},$$

$$\mu(1)=-2,\qquad\mu(1.3069297277...)=-1,\qquad\mu(1.6438001904...)=-\tfrac{1}{2},$$

$$\mu\left(\sqrt{3-\sqrt{6}}\right)=\mu(0.7419637843...)=-4,$$

$$\mu(2)=-0.2429928807...,\qquad\mu(3)=-0.0239463006...,$$

$$\mu\left(\sqrt{2}\right)=-0.7984598320...,\qquad\mu\left(2\sqrt{2}\right)=-0.0374612092....$$

The latter two values come from [52], where a different time scaling was chosen. Also, the constant $(3-6^{1/2})^{1/2}$ appears in [53, 54, 55] with regard to stopping rules in statistical sequential analysis.

For completeness' sake, here is the expression for $D_{-\lambda}(x)$ in terms of confluent hypergeometric functions:

$$D_{-\lambda}(x)=\frac{\sqrt{\pi}2^{-\lambda/2}}{\Gamma((1+\lambda)/2)}e^{-x^2/4}\Phi\left(\frac{\lambda}{2},\frac{1}{2},\frac{x^2}{2}\right)-2\frac{\sqrt{\pi}2^{-(1+\lambda)/2}}{\Gamma(\lambda/2)}xe^{-x^2/4}\Phi\left(\frac{1+\lambda}{2},\frac{3}{2},\frac{x^2}{2}\right)$$

which gives rise to the values $\lambda(1)$, $\lambda(2)$ and $\lambda^{-1}(-1/2)$ listed earlier. The constant $\mu^{-1}(-1)$ is important in the study of sample path behavior of Brownian motion [50, 56, 57] and first appeared in [54], as far as is known. Some higher dimensional results are given in [50, 58]. Csáki [59, 60] recently outlined the distributional asymptotics of the maximum M_T, but we cannot discuss this topic further.

0.5. Addendum. New numerical transform inversion algorithms [61, 62, 63] make enhancement of the tables in [25, 32] possible. Also, the distribution of the L_2-norm of X_t on $[0,T]$ can be inferred from closed-form expressions in [64, 65]. We wonder about corresponding results for L_1 and L_∞-norms. The conjectured formula for H_α in terms of the gamma function is probably false [66, 67, 68, 69]; simulation-based point estimates $H_{3/2}\approx0.77$ and confidence bounds $0.768\le H_{3/2}\le0.786$ do not carry over well to $H_{1/2}$ since the underlying algorithm becomes unreliable for $0<\alpha<1$.

REFERENCES

[1] G. E. Uhlenbeck and L. S. Ornstein, On the theory of Brownian motion, *Phys. Rev.* 36 (1930) 823–841.

[2] M. C. Wang and G. E. Uhlenbeck, On the theory of Brownian motion. II, *Rev. Modern Phys.* 17 (1945) 323–342; MR0013266 (7,130e).

[3] J. L. Doob, The Brownian movement and stochastic equations, *Annals of Math.* 43 (1942) 351–369; also in *Selected Papers on Noise and Stochastic Processes*, ed. N. Wax, Dover, 1954, pp. 319–337; MR0006634 (4,17d).

[4] L. Breiman, *Probability*, Addison-Wesley, 1968, pp. 347–351; MR0229267 (37 #4841).

[5] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, 1988, p. 358; MR1121940 (92h:60127).

[6] W. Feller, *An Introduction to Probability Theory and Its Applications*, v. 2, 2^{nd} ed., Wiley, 1971, pp. 96–99, 335–337; MR0270403 (42 #5292).

[7] I. I. Gringorten, Estimating finite-time maxima and minima of a stationary Gaussian Ornstein-Uhlenbeck process by Monte Carlo simulation, *J. Amer. Statist. Assoc.* 63 (1968) 1517–1521.

[8] L. Bartosch, Generation of colored noise, *Int. J. Mod. Phys. C* 12 (2001) 851–855.

[9] R. L. Honeycutt, Stochastic Runge-Kutta algorithms. I, White noise, *Phys. Rev. A* 45 (1992) 600–603; II, Colored noise, *Phys. Rev. A* 45 (1992) 604–610.

[10] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, 1992; MR1214374 (94b:60069) and MR1260431 (95a:60084).

[11] D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.* 43 (2001) 525–546; MR1872387 (2002j:65008).

[12] I. F. Blake and W. C. Lindsey, Level-crossing problems for random processes, *IEEE Trans. Inform. Theory* IT-19 (1973) 295–315; MR0370729 (51 #6955).

[13] J. Abrahams, A survey of recent progress on level-crossing problems for random processes, *Communications and Networks: A Survey of Recent Advances*, ed. I. F. Blake and H. V. Poor, Springer-Verlag, 1986, pp. 6–25.

[14] A. Göing-Jaeschke and M. Yor, A clarification note about hitting times densities for Ornstein-Uhlenbeck processes, *Finance Stochast.* 7 (2003) 413–415.

[15] L. Alili, P. Patie and J. L. Pedersen, Representations of the first hitting time density of an Ornstein-Uhlenbeck process, *Stoch. Models* 21 (2005) 967–980; MR2179308 (2006i:60111); http://www.math.ku.dk/~jesper/research/denou.html.

[16] M. U. Thomas, Some mean first-passage time approximations for the Ornstein-Uhlenbeck process, *J. Appl. Probab.* 12 (1975) 600–604; MR0388558 (52 #9394).

[17] S. Sato, Evaluation of the first-passage time probability to a square root boundary for the Wiener process, *J. Appl. Probab.* 14 (1977) 850–856; MR0471056 (57 #10797).

[18] L. M. Ricciardi and S. Sato, First-passage-time density and moments of the Ornstein-Uhlenbeck process, J. *Appl. Probab.* 25 (1988) 43–57; MR0929503 (89b:60189).

[19] S. R. Finch, Euler-Mascheroni constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 28–40.

[20] A. J. F. Siegert, On the first passage time probability problem, *Phys. Rev.* 81 (1951) 617–623; MR0054192 (14,887b).

[21] R. Bellman and T. Harris, Recurrence times for the Ehrenfest model, *Pacific J. Math.* 1 (1951) 179–193; MR0045323 (13,566b).

[22] D. A. Darling and A. J. F. Siegert, The first passage problem for a continuous Markov process, *Annals of Math. Statist.* 24 (1953) 624–639; MR0058908 (15,449f).

[23] J. A. Beekman, Gaussian-Markov processes and a boundary value problem, *Trans. Amer. Math. Soc.* 126 (1967) 29–42; MR0203793 (34 #3641).

[24] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, v. II, based on notes left by H. Bateman, McGraw-Hill, 1953, pp. 116–126; MR0058756 (15,419i).

[25] J. Keilson and H. F. Ross, Passage time distributions for Gaussian Markov (Ornstein-Uhlenbeck) statistical processes, *Selected Tables in Mathematical Statistics*, v. III, Amer. Math. Soc., 1975, pp. 233–327; MR0408175 (53 #11941).

[26] C. Park and F. J. Schuurmann, Evaluations of absorption probabilities for the Wiener process on large intervals, *J. Appl. Probab.* 17 (1980) 363–372; MR0568947 (81f:60111).

[27] K. Pötzelberger and L. Wang, Boundary crossing probability for Brownian motion, *J. Appl. Probab.* 38 (2001) 152–164; MR1816120 (2002a:60138).

[28] E. Di Nardo, A. G. Nobile, E. Pirozzi and L. M. Ricciardi, A computational approach to first-passage-time problems for Gauss-Markov processes, *Adv. Appl. Probab.* 33 (2001) 453–482; MR1842303 (2002e:60053).

[29] U. Sumita, J.-Y. Gotoh and H. Jin, Numerical exploration of dynamic behavior of Ornstein-Uhlenbeck processes via Ehrenfest process approximation, *J. Oper. Res. Soc. Japan* 49 (2006) 256–278; MR2267891 (2008g:60108).

[30] S. O. Rice, Distribution of the duration of fades in radio transmission: Gaussian noise model, *Bell System Tech. J.* 37 (1958) 581–635; MR0094269 (20 #788).

[31] D. Slepian, The one-sided barrier problem for Gaussian noise, *Bell System Tech. J.* 41 (1962) 463–501; MR0133183 (24 #A3017).

[32] J. Keilson and H. F. Ross, *Gaussian Markov Related Variates for Meteorological Planning*, AFGL-TR-79-0282, ADA081382 (1979), Univ. of Rochester; available through the National Technical Information Service, Springfield VA 22161.

[33] D. M. DeLong, Crossing probabilities for a square root boundary by a Bessel process, *Comm. Statist. A–Theory Methods* 10 (1981) 2197–2213; erratum 12 (1983) 1699; MR0629897 (82i:62119) and MR0711257 (84i:62106).

[34] J.-M. Azaïs and M. Wschebor, The distribution of the maximum of a Gaussian process: Rice method revisited, *In and Out of Equilibrium: Probability with a Physics Flavor,* Proc. 2000 Mambucaba conf., ed. V. Sidoravicius, Birkhäuser Boston, 2002, pp. 321–348; MR1901961 (2003d:60075).

[35] A. Estrella, Critical values and p values of Bessel process distributions: Computation and application to structural break tests, *Econometric Theory* 19 (2003) 1128–1143.

[36] G. Louchard, C. Kenyon and R. Schott, Data structures' maxima, *SIAM J. Comput.* 26 (1997) 1006–1042; MR1460714 (98m:68045).

[37] J. Pickands, Upcrossing probabilities for stationary Gaussian processes, *Trans. Amer. Math. Soc.* 145 (1969) 51–73; MR0250367 (40 #3606).

[38] J. Pickands, Asymptotic properties of the maximum in a stationary Gaussian process, *Trans. Amer. Math. Soc.* 145 (1969) 75–86; MR0250368 (40 #3607).

[39] S. M. Berman, Maxima and high level excursions of stationary Gaussian processes, *Trans. Amer. Math. Soc.* 160 (1971) 65–85; MR0290449 (44 #7630).

[40] C. Qualls and H. Watanabe, Asymptotic properties of Gaussian processes, *Annals of Math. Statist.* 43 (1972) 580–596; MR0307318 (46 #6438).

[41] M. R. Leadbetter, G. Lindgren and H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, 1983, pp. 216–237; MR0691492 (84h:60050).

[42] S. R. Finch, Extreme value constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 363–367.

[43] Q.-M. Shao, Bounds and estimators of a basic constant in extreme value theory of Gaussian processes, *Statist. Sinica* 6 (1996) 245–257; MR1379060 (97e:60048).

[44] K. Debicki, Z. Michna and T. Rolski, Simulation of the asymptotic constant in some fluid models, *Stoch. Models* 19 (2003) 407–423; MR1993949.

[45] W. V. Li and Q.-M. Shao, Gaussian processes: Inequalities, small ball probabilities and applications, *Stochastic Processes: Theory and Methods*, ed. D. N. Shanbhag and C. R. Rao, North-Holland, 2001, pp. 533–597; MR1861734.

[46] P. Mandl, The distribution of time needed for the Uhlenbeck process to reach a border (in Czech), *Aplikace Mat.* 7 (1962) 141–148; Zbl. 0138.11303.

[47] P. Mandl, *Analytical Treatment of One-Dimensional Markov Processes*, Springer-Verlag, 1968, p. 145; MR0247667 (40 #930).

[48] J. A. Beekman, Asymptotic distributions for the Orstein-Uhlenbeck process, *J. Appl. Probab.* 12 (1975) 107–114; MR0388521 (52 #9357).

[49] K. Uchiyama, Brownian first exit from and sojourn over one-sided moving boundary and application, *Z. Wahrsch. Verw. Gebiete* 54 (1980) 75–116; MR0595482 (82c:60143).

[50] E. Perkins, On the Hausdorff dimension of the Brownian slow points, *Z. Wahrsch. Verw. Gebiete* 64 (1983) 369–399; MR0716493 (85g:60085b).

[51] L. Breiman, First exit times from a square root boundary, *Proc. Fifth Berkeley Symp. Math. Stat. Probab.*, vol. II, part 2, ed. L. M. Le Cam and J. Neyman, Univ. of Calif. Press, 1967, pp. 9–16; MR0212865 (35 #3730).

[52] K. Lindenberg, K. E. Shuler, J. Freeman and T. J. Lie, First passage time and extremum properties of Markov and independent processes, *J. Statist. Phys.* 12 (1975) 217–251; MR0458604 (56 #16804).

[53] Y. S. Chow and H. Teicher, On second moments of stopping rules, *Annals of Math. Statist.* 37 (1966) 388–392; MR0192528 (33 #753).

[54] L. A. Shepp, A first passage problem for the Wiener process, *Annals of Math. Statist.* 38 (1967) 1912–1914; MR0217879 (36 #968).

[55] B. M. Brown, Moments of a stopping rule related to the central limit theorem, *Annals of Math. Statist.* 40 (1969) 1236–1249; MR0243689 (39 #5010).

[56] B. Davis, On Brownian slow points, *Z. Wahrsch. Verw. Gebiete* 64 (1983) 359–367; MR0716492 (85g:60085a).

[57] P. Greenwood and E. Perkins, A conditioned limit theorem for random walk and Brownian local time on square root boundaries, *Annals of Probab.* 11 (1983) 227–261; MR0690126 (85d:60068).

[58] F. Haake, J. W. Haus and R. Glauber, Passage-time statistics for the decay of unstable equilibrium states, *Phys. Rev. A* 23 (1981) 3255–3271; MR0615182 (82f:81039).

[59] E. Csáki and Z. Shi, Some lim inf results for two-parameter processes, *Stochastic Process. Appl.* 78 (1998) 27–46; MR1653288 (99i:60064).

[60] E. Csáki and Y. Hu, On the ranked excursion heights of a Kiefer process, *J. Theoret. Probab.* 17 (2004) 145–163; MR2054583.

[61] P. Valkó and J. Abate, Numerical Laplace Inversion (2002), http://library.wolfram.com/infocenter/MathSource/4738/.

[62] J. Abate and P. Valkó, Numerical Inversion of Laplace Transform with Multiple Precision Using the Complex Domain (2003), http://library.wolfram.com/infocenter/MathSource/5026/.

[63] J. Abate and P. Valkó, Multi-precision Laplace transform inversion, *Internat. J. Numer. Methods Engineering.* 60 (2004) 979–993; http://www.pe.tamu.edu/valko/public_html/Nil/.

[64] T. Dankel, On the distribution of the integrated square of the Ornstein-Uhlenbeck process, *SIAM J. Appl. Math.* 51 (1991) 568–574; MR1095035 (92c:60056).

[65] F. Gao, J. Hannig, T.-Y. Lee and F. Torcaso, Laplace transforms via Hadamard factorization, *Elec. J. Probab.* 8 (2003) P13; MR1998764 (2005h:60110).

[66] K. Burnecki and Z. Michna, Simulation of Pickands constants, *Probab. Math. Statist.* 22 (2002) 193–199; MR1944151.

[67] K. Debicki and P. Kisowski, A note on upper estimates for Pickands constants, *Statist. Probab. Lett.* 78 (2008) 2046–2051; MR2458013 (2009k:60089).

[68] A. J. Harper, Bounds on the suprema of Gaussian processes, and omega results for the sum of a random multiplicative function, *Annals Appl. Probab.* 23 (2013) 584–616; MR3059269.

[69] A. B. Dieker and B. Yakir, On asymptotic constants in the theory of extremes for Gaussian processes, *Bernoulli* 20 (2014) 1600–1619; arXiv:1206.5840; MR3217455.