Gergonne-Schwarz Surface

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We mentioned Plateau's problem in [1] but did not give a nontrivial example. Let

$$F[\phi, m] = \int_{0}^{\sin(\phi)} \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-m\,\tau^2}}$$

denote the incomplete elliptic integral of the first kind and $K[m] = F[\pi/2, m]$; the latter is admittedly incompatible with [2] but we purposefully choose formulas here to be consistent with the computer algebra package MATHEMATICA. The three basic Jacobi elliptic functions are defined via

$$\begin{aligned} u &= \int_{0}^{\mathrm{sn}(u,m)} \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-m\,\tau^2}} = \int_{\mathrm{cn}(u,m)}^{1} \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{m\,\tau^2+(1-m)}} \\ &= \int_{\mathrm{dn}(u,m)}^{1} \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{\tau^2-(1-m)}} \end{aligned}$$

and two (of nine) others we require are

$$\mathrm{sc}(u,m) = \frac{\mathrm{sn}(u,m)}{\mathrm{cn}(u,m)}, \qquad \mathrm{sd}(u,m) = \frac{\mathrm{sn}(u,m)}{\mathrm{dn}(u,m)}.$$

Our work supplements [3] very closely, even down to the level of notation. The setting is three-dimensional xyz-space.

0.1. Six Edges of a Cube. Consider a polygonal wire loop with six line segments:

$$(0,0,0) \to (1,0,0) \to (1,0,1) \to (1,1,1) \to (0,1,1) \to (0,1,0) \to (0,0,0).$$

What is the minimal area for any surface spanning this fixed boundary? Equivalently, what is the outcome of dipping the wire loop in a soap solution?

Define

$$\rho_0 = K[1/4] = 1.6857503548...$$

and let $t = \mathcal{E}(\xi)$ denote the functional inverse of the elliptic integral

$$\xi = \int_0^t \frac{d\tau}{\sqrt{1+\tau^2+\tau^4}}.$$

The desired minimal surface is given implicitly by the equation [3]

$$\mathcal{E}(x)\mathcal{E}(y) = \mathcal{E}(z)$$

where $0 \le x, y, z \le \rho_0$.

This is as far as Nitsche [3] went in describing his calculations. Solving for z and rescaling (so that the surface spans the $1 \times 1 \times 1$ cube), we find that

$$z = \frac{1}{2\rho_0} F\left[\arccos\left(\frac{\operatorname{cn}\left(2\rho_0 x, \frac{1}{4}\right) + \operatorname{cn}\left(2\rho_0 y, \frac{1}{4}\right)}{1 + \operatorname{cn}\left(2\rho_0 x, \frac{1}{4}\right)\operatorname{cn}\left(2\rho_0 y, \frac{1}{4}\right)}\right), \frac{1}{4}\right], \qquad 0 \le x, y \le 1$$

and the surface area is

$$2\int_0^1 \int_0^{1-x} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dy\, dx = \frac{3}{2}\frac{K[3/4]}{K[1/4]} = 1.9188923567...,$$

as predicted in [4]. See Figure 1.

0.2. Four Edges of a Regular Tetrahedron. Consider a polygonal wire loop with four line segments:

$$(0,0,0) \to (1,0,1) \to (1,1,0) \to (0,1,1) \to (0,0,0).$$

Again, what is the minimal area for any surface spanning this fixed boundary?

With ρ_0 as before, let $s = \mathcal{F}(\eta)$ denote the functional inverse of the elliptic integral

$$\eta = \int_0^s \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}}.$$

The desired minimal surface is given implicitly by the equation [3]

$$\mathcal{F}(y)\mathcal{F}(z) + \mathcal{F}(z)\mathcal{F}(x) + \mathcal{F}(x)\mathcal{F}(y) + 1 = 0$$

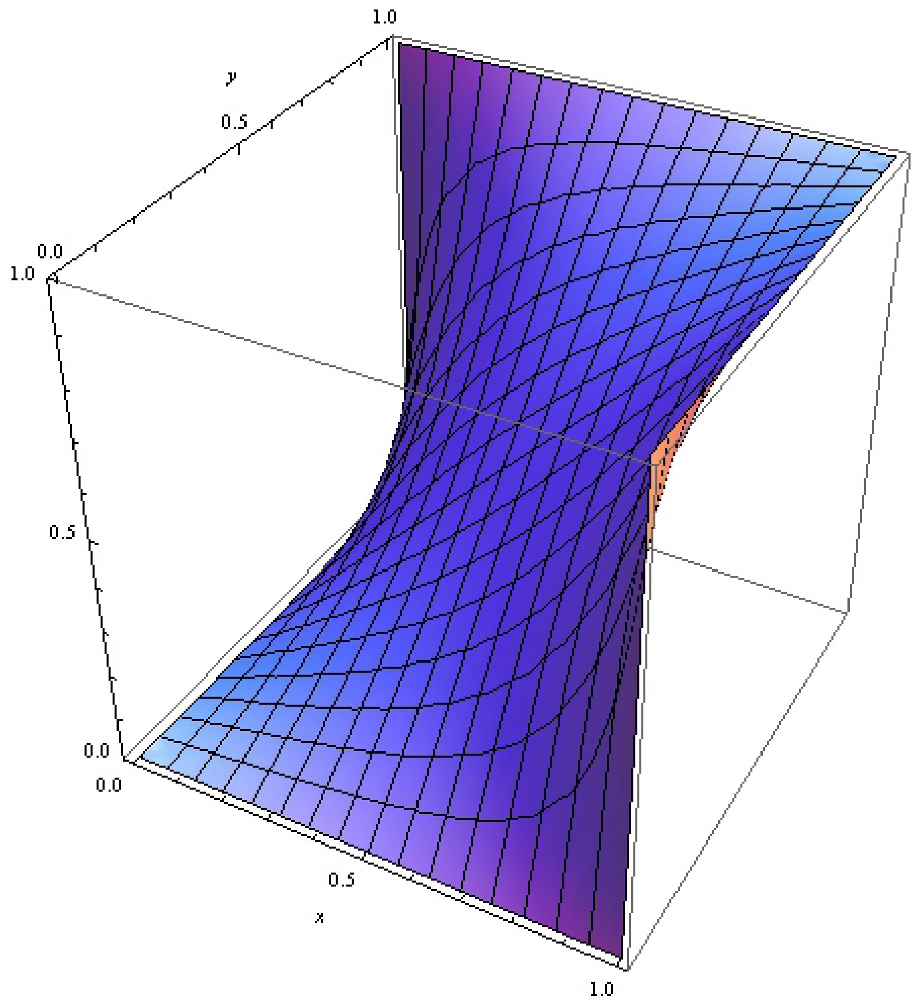

Figure 1: “Six edges” minimal surface

where $0 \leq x, y \leq \rho_0$ and $-\rho_0 \leq z \leq 0$. Dalpe [5] introduced one correction in the preceding: the cube has side ρ_0, not $2\rho_0$.

This is as far as described in [3]. Solving for z and rescaling (so that the surface spans the $1 \times 1 \times 1$ cube), we find that

$$z = \frac{1}{\sqrt{3}\rho_0} F\left[\arccos\left(\frac{\operatorname{cn}\left(\sqrt{3}\rho_0 x, -\frac{1}{3}\right)\operatorname{cn}\left(\sqrt{3}\rho_0 y, -\frac{1}{3}\right)}{1+\operatorname{sn}\left(\sqrt{3}\rho_0 x, -\frac{1}{3}\right)\operatorname{sn}\left(\sqrt{3}\rho_0 y, -\frac{1}{3}\right)}\right), -\frac{1}{3}\right], \qquad 0 \leq x, y \leq 1$$

(note multiplication in the numerator and sn in the denominator, unlike before) and the surface area is

$$2\int_0^1 \int_0^{1-x} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dy\, dx = \frac{K[3/4]}{K[1/4]} = 1.2792615711...,$$

as predicted in [4]. See Figure 2. This example and the first one feature portions of what is known as the *Schwarz D surface* (D stands for "Diamond").

0.3. Two Diagonals and Free Boundaries. Consider the soap film (resembling a twisted curtain) formed between two skew line segments:

$$(2,0,0) \to (0,2,0) \qquad \text{and} \qquad (0,0,2) \to (2,2,2).$$

Understanding that two remaining boundaries are unspecified, what is the minimal area for any surface spanning the diagonals? [6] This is a famous question due to Gergonne (1816) and answered by Schwarz (1872).

For fixed $\kappa > 0$, let $t = Q(\varphi, \kappa)$ and $t = R(\psi, \kappa)$ denote functional inverses of the elliptic integrals

$$\varphi = \int_0^t \frac{d\tau}{\sqrt{\kappa - \tau^2 - \tau^4}}, \qquad \psi = \int_0^t \frac{d\tau}{\sqrt{\kappa + (1+2\kappa)\tau^2 + \kappa\tau^4}}.$$

Define also

$$\lambda(\kappa) = \frac{\sqrt{1+4\kappa}-1}{2\sqrt{1+4\kappa}}, \qquad \mu(\kappa) = \sqrt{\frac{\sqrt{1+4\kappa}-1}{2}}.$$

We have, in particular,

$$\int_0^{\mu(\kappa)} \frac{d\tau}{\sqrt{\kappa - \tau^2 - \tau^4}} = \frac{K[\lambda(\kappa)]}{(1+4\kappa)^{1/4}},$$

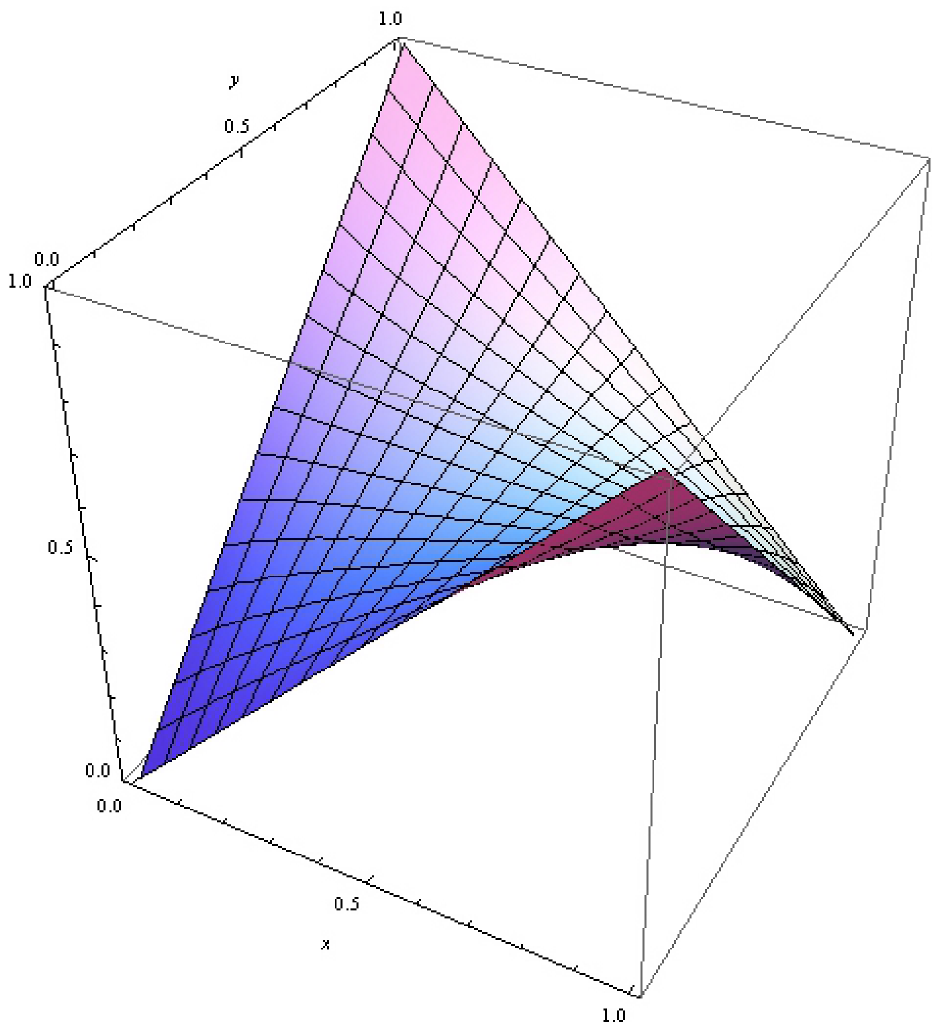

Figure 2: Tetrahedral “four edges” minimal surface

$$\int_0^1 \frac{d\tau}{\sqrt{\kappa + (1+2\kappa)\tau^2 + \kappa\,\tau^4}} = \frac{K\left[-\frac{1}{4\kappa}\right]}{2\sqrt{\kappa}}$$

and these two expressions, when set equal, force $\kappa = \kappa_0 = 0.2092861374....$ Denote the former integral by φ_0 and latter by ψ_0; consequently $\varphi_0 = \psi_0 = 1.3970394887....$ The desired minimal surface is given implicitly by the equation [3]

$$Q(x-\varphi_0)R(z-\psi_0) + Q(y-\varphi_0) = 0$$

where $0 \le x, y \le 2\varphi_0$ and $0 \le z \le 2\psi_0$. We have introduced two corrections in the preceding: the upper integration limit of ψ_0 is 1 (not $\mu(\kappa)$, which was a typographical error in [3]) and the denominator underlying $K\left[-\frac{1}{4\kappa}\right]$ is $2\sqrt{\kappa}$ (not merely 2, which was a computational error in [3]). More on the second correction will be mentioned shortly.

This, again, is as far as described in [3]. Let

$$\theta_0 = (1+4\kappa_0)^{1/4}\,\varphi_0, \qquad \lambda_0 = \lambda(\kappa_0), \qquad \varepsilon(x,y) = \begin{cases} 1 & \text{if } (x-1)(y-1) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

Solving for z and rescaling (so that the surface spans the $2\times2\times2$ cube), we find that

$$z = 1 + \frac{\varepsilon(x,y)}{2\sqrt{\kappa}\,\psi_0} F\left[\arccos\left(\frac{\operatorname{sd}\left(\theta_0(x-1),\lambda_0\right)^2 - \operatorname{sd}\left(\theta_0(y-1),\lambda_0\right)^2}{\operatorname{sd}\left(\theta_0(x-1),\lambda_0\right)^2 + \operatorname{sd}\left(\theta_0(y-1),\lambda_0\right)^2}\right), -\frac{1}{4\kappa_0}\right]$$

assuming ($y > x$ and $x < 2 - y$) or ($y < x$ and $x > 2 - y$); elsewhere on $0 \le x, y \le 2$, no definition for z is given. The surface area is

$$4\int_0^1 \int_0^{1-x} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dy\, dx = 4.9348196582... = 4\,(1.2337049145...)$$

and a closed-form expression remains open. See Figure 3. We have not attempted to establish consistency with [7].

0.4. Details of Elliptic Functions. We can compute $\mathcal{E}(\xi)$ and $\mathcal{F}(\eta)$ using results in [8]:

$$\xi = \int_0^t \frac{d\tau}{\sqrt{1+\tau^2+\tau^4}} = \frac{1}{2}F\left[\arccos\left(\frac{1-t^2}{1+t^2}\right), \frac{1}{4}\right],$$

$$\eta = \int_0^s \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} = \frac{1}{\sqrt{3}}F\left[\arccos\left(\frac{1-s^2}{1+s^2}\right), -\frac{1}{3}\right]$$

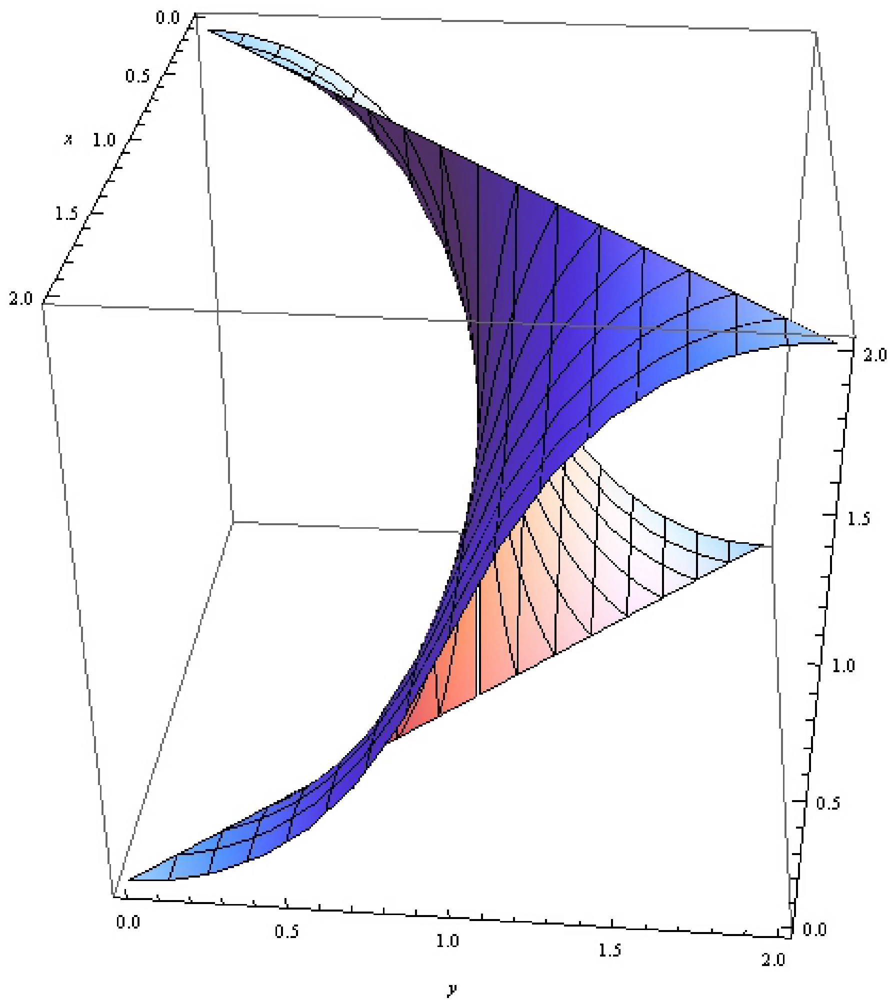

Figure 3: "Two diagonals" minimal surface

since each quartic has four imaginary zeroes; hence

$$t = \sqrt{\frac{1 - \mathrm{cn}\,(2\xi, 1/4)}{1 + \mathrm{cn}\,(2\xi, 1/4)}},$$

$$s = \sqrt{\frac{1 - \mathrm{cn}\left(\sqrt{3}\eta, -1/3\right)}{1 + \mathrm{cn}\left(\sqrt{3}\eta, -1/3\right)}}$$

and thus

$$z = \frac{1}{2} F\left[\arccos\left(\frac{1 - \mathcal{E}(x)^2\mathcal{E}(y)^2}{1 + \mathcal{E}(x)^2\mathcal{E}(y)^2}\right), \frac{1}{4}\right]$$

gives the "six edges" result. From

$$\mathcal{F}(z) = -\frac{1 + \mathcal{F}(x)\mathcal{F}(y)}{\mathcal{F}(x) + \mathcal{F}(y)}$$

we obtain

$$z = \frac{1}{\sqrt{3}} F\left[\arccos\left(\frac{1 - \left(\frac{1+\mathcal{F}(x)\mathcal{F}(y)}{\mathcal{F}(x)+\mathcal{F}(y)}\right)^2}{1 + \left(\frac{1+\mathcal{F}(x)\mathcal{F}(y)}{\mathcal{F}(x)+\mathcal{F}(y)}\right)^2}\right), -\frac{1}{3}\right]$$

and, because $\mathrm{sn}(u, m)^2 + \mathrm{cn}(u, m)^2 = 1$, the "four edges" result follows.

Computing $Q(\varphi, \kappa)$ is somewhat different [9]:

$$\begin{aligned}
\varphi &= \int_0^t \frac{d\tau}{\sqrt{\kappa - \tau^2 - \tau^4}} \\
&= \frac{1}{(1+4\kappa)^{1/4}} \left\{ K[\lambda(\kappa)] - F\left[\arcsin\left(\sqrt{\frac{\sqrt{1+4\kappa} - 2t^2 - 1}{\sqrt{1+4\kappa} - 1}}\right), \lambda(\kappa)\right] \right\}
\end{aligned}$$

since the quartic has two real zeroes and two imaginary zeroes. Observe that, when $t = \mu(\kappa)$, the second term vanishes. Inverting, we obtain

$$t = \frac{\kappa}{(1+4\kappa)^{1/4}} \,\mathrm{sd}\left((1+4\kappa)^{1/4}\varphi, \lambda(\kappa)\right)$$

and therefore

$$-\frac{Q(y - \varphi_0, \kappa)}{Q(x - \varphi_0, \kappa)} = -\frac{\mathrm{sd}\left((1+4\kappa)^{1/4}(y - \varphi_0), \lambda(\kappa)\right)}{\mathrm{sd}\left((1+4\kappa)^{1/4}(x - \varphi_0), \lambda(\kappa)\right)}.$$

Only the inverse of $R(\psi,\kappa)$ is required:

$$\psi=\int_0^t \frac{d\tau}{\sqrt{\kappa+(1+2\kappa)\tau^2+\kappa\,\tau^4}}=\frac{\operatorname{sign}(t)}{2\sqrt{\kappa}}F\left[\arccos\left(\frac{1-t^2}{1+t^2}\right),-\frac{1}{4\kappa}\right]$$

which generalizes the earlier cases $\kappa=-1$ and $\kappa=3/4$. Note the specialization $t=1$, as well as the need here to track whether $t=-Q(y-\varphi_0,\kappa)/Q(x-\varphi_0,\kappa)$ is positive or negative.

0.5. Approximations of Minimal Surfaces. A surprisingly good fit to the "four edges" surface is provided by the hyperbolic paraboloid

$$z=x+y-2xy$$

and the corresponding surface area is

$$2\int_0^1\int_0^{1-x}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dy\,dx=1.2807...>1.2792....$$

See [10] for more on approximating the Schwarz D surface, which (upon suitable transformation) should enable a reasonable fit to the "six edges" surface.

Fairly coarse fits to the "two diagonals" surface are provided by

$$z=1+\frac{y-1}{x-1},\qquad z=1+\frac{4}{\pi}\arctan\left(\frac{y-1}{x-1}\right)$$

if ($y>x$ and $x<2-y$) or ($y<x$ and $x>2-y$), and the corresponding surface areas are 5.1231... and 5.0307..., respectively. We mentioned earlier that Nitsche [3] mistakenly solved the equation

$$\frac{K[\lambda(\kappa)]}{(1+4\kappa)^{1/4}}=\frac{K\left[-\frac{1}{4\kappa}\right]}{2};$$

the denominator underlying $K\left[-\frac{1}{4\kappa}\right]$ is missing a factor $\sqrt{\kappa}$. It is nevertheless instructive to follow through to the end. We find $\kappa=\widetilde{\kappa}_0=6.6061877190...$ and consequently $\widetilde{\varphi}_0=\widetilde{\psi}_0=0.7781217795....$ The surface obtained *is* a minimal surface (with mean curvature everywhere equal to zero) and correctly spans the diagonals. The two free contours, however, are not best possible: the surface area for $\widetilde{\kappa}_0$ is 4.9480..., which is larger than the surface area 4.9348... for κ_0.

The constant 1.9188... appears in [11, 12], 1.2792... in [13, 14] and a rough estimate for $\frac{1}{4}(4.9348...)$ in [15]. See [16, 17] for introductory materials, as well as Schwarz's complete works [18]. Other polygonal wire loops, with more solutions of Plateau's problem, are surveyed in [19].

0.6. Acknowledgements. Ulrike Bücking was so kind as to point out two errors in [3]; I also appreciate correspondence with Ðjurđje Cvijović and Stefan Hildebrandt.

0.7. Addendum. Another portion of the Schwarz D surface arises as a soap film spanning two parallel equilateral triangles with vertices

$$\{(1,-1,-1),(-1,1,-1),(-1,-1,1)\} \qquad \text{and} \qquad \{(-1,1,1),(1,-1,1),(1,1,-1)\}.$$

One triangle is a copy of the other, rotated 60° about its center. Each of the six edges has length $2\sqrt{2}$ and the perpendicular distance between triangular centers is $2/\sqrt{3}$; the ratio of these is $\sqrt{6}$. Define $\zeta_0 = K[8/9]$. The desired minimal annulus is given implicitly by [18, 20]

$$\operatorname{sc}(\zeta_0 y, \tfrac{8}{9})\operatorname{sc}(\zeta_0 z, \tfrac{8}{9}) + \operatorname{sc}(\zeta_0 z, \tfrac{8}{9})\operatorname{sc}(\zeta_0 x, \tfrac{8}{9}) + \operatorname{sc}(\zeta_0 x, \tfrac{8}{9})\operatorname{sc}(\zeta_0 y, \tfrac{8}{9}) + 3 = 0$$

where $-1 \le x, y, z \le 1$ and its surface area is $6K[3/4]/K[1/4]$. See Figure 4. (This result contradicts a statement in [21] that, for Schwarz D to appear, the ratio of edge length to distance should be $2\sqrt{3}$.)

A more difficult task is to represent the minimal annulus corresponding to parallel triangles that are aligned [22, 23, 24, 25, 26], that is, with no rotation. This is a member of the family of *Schwarz H surfaces* (H stands for "Hexagonal"). Assistance on such representations, for a range of perpendicular distances between triangular centers, and on numerical calculation of surface areas, would be deeply appreciated.

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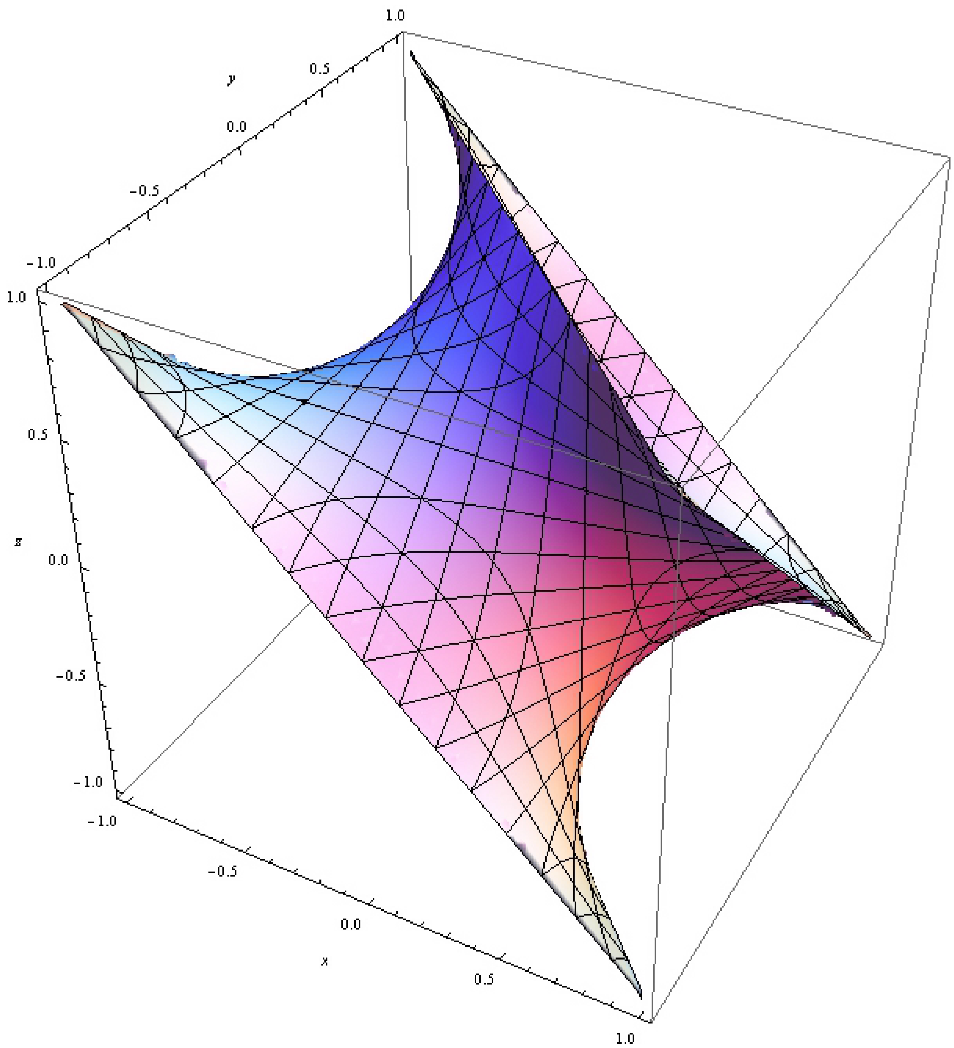

Figure 4: "Two twisted triangles" minimal surface

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