Gol'dberg's Zero-One Constants

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May 21, 2014

Let F be the set of all functions f that are analytic on some ring $\{z : \rho(f) < |z| < 1\}$ and omit the values of both 0 and 1 there. Each function is defined in its own distinct ring. By *omit*, it is meant that $f(z) \notin \{0, 1\}$ for all z. We assume $\rho(f)$ to be as small as possible. Let $G \subseteq F$ consist of all functions that are analytic on the open unit disk D. Thus, for $f \in G$, we have

$$\rho(f) = \begin{cases} 0 & \text{if } f \text{ is never 0 or 1,} \\ \sup\{|z| : \ f(z) \in \{0,1\}\} & \text{otherwise.} \end{cases}.$$

Given a real number a, the **a-points** of f are the points z for which $f(z) = a$. Of course, 0-points are more commonly referred to as *zeroes*.

Consider the circle σ defined by

$$\left\{z : |z| = \sqrt{\rho(f)}\right\}$$

with counterclockwise orientation, and let γ_f be the image of σ under f. The **index** (or **winding number**) of γ_f with respect to the point a is

$$n(\gamma_f, a) = \frac{1}{2\pi i} \int\limits_{\gamma_f} \frac{dz}{z-a}.$$

Our interest is in the scenario when $n(\gamma_f, 0)$, $n(\gamma_f, 1)$ are nonzero and distinct; without loss of generality, we assume that $n(\gamma_f, 0) > n(\gamma_f, 1)$. Let $F(N_0, N_1) \subseteq F$ consist of all functions f with $n(\gamma_f, 0) = N_0$ and $n(\gamma_f, 1) = N_1$. Let $G(M_0, M_1) \subseteq G$ consist of all functions g with exactly one 0-point [of multiplicity M_0] and exactly one 1-point [of multiplicity M_1]. Again, we focus on $M_0 \neq 0$, $M_1 \neq 0$ and $M_0 \neq M_1$; without loss of generality, assume that $M_0 > M_1$.

Gol'dberg [1] studied constants similar to

$$A(N_0, N_1) = \inf\{\rho(f) : f \in F(N_0, N_1)\},$$

$$B(M_0, M_1) = \inf\{\rho(g) : g \in G(M_0, M_1)\}.$$

Bergweiler & Eremenko [2] discovered closed-form expressions:

$$A(2,1)=\nu=\exp\left(-\frac{\pi^2}{\ln\left(3+2\sqrt{2}\right)}\right)=0.0037015991...,$$

$$A(3,1)=A(3,2)=\exp\left(-\frac{\pi^2}{\ln\left(5+2\sqrt{6}\right)}\right)=0.0134968456...,$$

$$A(4,1)=A(4,3)=\exp\left(-\frac{\pi^2}{\ln\left(7+4\sqrt{3}\right)}\right)=0.0235855221...$$

and moreover proved that

$$A=\inf\left\{\rho(f):f\in F\text{ and }N_0>N_1\geq 1\right\}=\nu.$$

(Gol'dberg's original bounds for A were strengthened by Jenkins [3].) The numerical computation of

$$B(2,1)=\mu=0.0252896...,$$

$$B(3,1)=0.084924..., \qquad B(3,2)=0.227417...,$$

$$B(4,1)=0.140571..., \qquad B(4,3)=0.290697...$$

is more difficult – no precise formulas are known – and it is merely conjectured that

$$B=\inf\left\{\rho(g):g\in G\text{ and }M_0>M_1\geq 1\right\}=\mu.$$

(The best lower bound 0.00587 for B in [2], improving on [4, 5, 6], is still far off.) An elaborate construction of a certain transcendental analytic function on D possessing exactly one 0-point at $-\mu$ [with $M_0=2$] and exactly one 1-point at μ [with $M_1=1$] occupies much of the discussion in [2]. It shows that $B\leq\mu$. A proof that $B\geq\mu$ remains open.

0.1. Belgian Chocolate Problem. Here the difficulties of construction are overwhelming. What is the smallest $\tau>0$ for which there exists an analytic function on D possessing exactly one 0-point at 0 [of multiplicity 1] and exactly two 1-points at $\pm\tau$ [each of multiplicity 1]? The current best bounds are [2, 7]

$$0.01450779<\tau<0.10913022.$$

Blondel's question [8, 9] is often phrased as follows. Let $a(z)=z^2-2\delta z+1$ and $b(z)=z^2-1$. What is the largest $\delta>0$ for which there exist stable real polynomials p and q with $\deg(p)\geq\deg(q)$ such that $ap+bq$ is stable? (A polynomial is called

stable if all its zeroes are in the left half plane.) The numbers τ and δ are related by

$$\tau = \sqrt{\frac{1-\delta}{1+\delta}}, \qquad \delta = \frac{1-\tau^2}{1+\tau^2}$$

and the current best bounds are

$$0.97646152 < \delta < 0.99957913.$$

Incremental progress in specifying such constraints is found in [4, 10, 11, 12, 13, 14].

0.2. Landau's Theorem with Explicit Bound. If an analytic function g on D omits the values of both 0 and 1, then [15, 16, 17]

$$|g(0)| \leq 2\,|g'(0)|\,(|\ln|g(0)|| + K)$$

where the constant

$$K = \frac{1}{4\pi^2}\Gamma\left(\frac{1}{4}\right)^4 = 4.3768792304...$$

is best possible. Other occurrences of K are similar to results appearing in [18]. If analytic g satisfies $g(0) = 0$ and $g'(0) = 1$, then $g(D)$ covers a segment of each line passing through the origin; further, each segment has length at least $2/K = 0.4569465810...$ and this is sharp [19, 20, 21]. If analytic g satisfies $g(-z) = -g(z)$ for all $z \in D$ and $g'(0) = 1$, then $g(D)$ covers a disk with center at the origin and radius $1/K = 0.2284732905...$; again, this is sharp [8, 22]. The presence of the elliptic modular function

$$J(z) = 16\exp(\pi i z)\prod_{n=1}^{\infty}\left(\frac{1+\exp(2n\pi i z)}{1+\exp((2n-1)\pi i z)}\right)^8, \qquad \operatorname{Im}(z) > 0$$

$$\frac{1}{J'(i)} = \frac{4}{K}i$$

is keenly felt here.

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