Another Special Rational Sequence

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In "The Special Rational Sequence", I talked about the following mapping:

$$\upsilon : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{Q}, n \longmapsto \cfrac{1}{2 \times \cfrac{2}{2 \times \cfrac{3}{2 \times \cfrac{4}{\vdots}}}} = \left(\frac{(n-1)!!}{2^{\frac{1+(-1)^n}{2}} \cdot n!!} \right)^{(-1)^n}$$

However, there is another sequence that is created in a similar vein, albeit in the opposing direction.

Problem

Find the function for the value

$$\cfrac{2 \times \cfrac{2 \times \cfrac{2 \times \cfrac{\vdots}{4}}{3}}{2}}{1}$$

Solution

Firstly, let's approach this in a form that's familiar to us:

$$\cfrac{2 \times \cfrac{2 \times \cfrac{2 \times \cfrac{\vdots}{4}}{3}}{2}}{1} = (2(2(2(\cdots 2(2n)(n-1)^{-1} \cdots)4^{-1})3^{-1})2^{-1})1^{-1} = f(1,n)$$

where

$$f : (\mathbb{N} \setminus \{0\})^2 \longrightarrow \mathbb{Q}, (m,n) \longmapsto \begin{cases} m & \text{if } m = n \\ \dfrac{2f(m+1,n)}{m} & \text{otherwise} \end{cases}$$

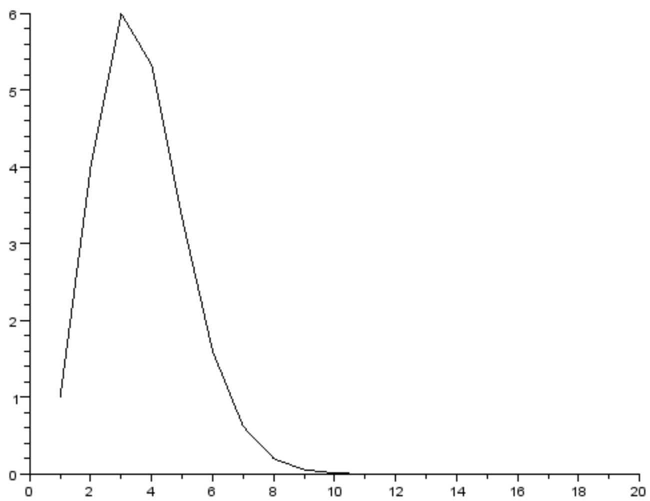
Straight off the bat, we notice that the denominator can be simplified to the factorial $(n-1)!$. In addition, because there is a 2 for every number present in the denominator, we can simplify the product of 2's to a single exponential (2^{n-1}).

So now, we can reduce our pattern down to a single function, ν:[1]

$$\nu : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{Q}, n \longmapsto \frac{n2^{n-1}}{(n-1)!}$$

Further Analysis

The first thing about this rational sequence–and one that differentiates it from the first sequence–is that it is definitely convergent, namely towards zero.



The graph above was created using the Scilab program below.[2] As can be observed from the graph, the sequence starts to converge at $n = 3$.

```
function y = eta(n)
    if isreal(n) & (int(n) == n) & (n > 0) then
        y = n .* 2.^(n - 1)./factorial(n - 1);
    end
endfunction
```

But is there a relationship between the upsilon numbers and the nu numbers? Well, when n is odd (i.e. $n \in \mathbb{N} \setminus 2\mathbb{N}$), $\upsilon_n = \dfrac{n!!}{(n-1)!!}$, whereas when n is even (i.e. $n \in 2\mathbb{N} \setminus \{0\}$), $\upsilon_n = \dfrac{(n-1)!!}{2n!!}$.

When $n \in \mathbb{N} \setminus 2\mathbb{N}$

$$\upsilon_n \nu_n = \frac{n!!}{(n-1)!!} \cdot \frac{n2^{n-1}}{(n-1)!}$$

[1] Like upsilon (υ) above, the choice of nu (ν) is purely aesthetic. Becuase of its relation to the 'upsilon rationals', I would be inclined to refer to these as 'upsilon numbers of the second kind' (whilst referring to the other upsilon numbers as 'upsilon numbers of the first kind'), but I'm easy whichever way people choose.

[2] Again, the program has been written with the intention of being usable with a matrix of values. However, unlike `upsilon`, which produces an error when the parameters are wrong, this just doesn't return anything, which is a little dangerous.

$$
\begin{aligned}
&= \frac{n!!n2^{n-1}}{(n-1)!!(n-1)!} \\
&= \frac{n^2 2^{n-1}(n-2)!!}{(n-1)!!(n-1)!} \\
&= \frac{n^2 2^{n-1}}{[(n-1)!!]^2}
\end{aligned}
$$

$$
\begin{aligned}
\upsilon_n \nu_n &= \frac{(n-1)!!n2^{n-1}}{2n!!(n-1)!} \\
&= \frac{n2^{n-2}}{n!!(n-2)!!} \\
&= \frac{2^{n-2}}{[(n-2)!!]^2}
\end{aligned}
$$

From this observation, we see that, $\forall n \in \mathbb{N} \setminus 2\mathbb{N},\ \upsilon_n \nu_n = n^2 \upsilon_{n+1} \nu_{n+1}$–or

$$
\frac{\upsilon_n \nu_n}{\upsilon_{n+1} \nu_{n+1}} = n^2 \ \forall n \in \mathbb{N} \setminus 2\mathbb{N}
$$

Values of υ_n and ν_n for $n \in \mathbb{N} \cap [1, 20]$

n	υ_n	ν_n
1	1	1
2	$\frac{1}{4}$	4
3	$\frac{3}{2}$	6
4	$\frac{3}{16}$	$\frac{16}{3}$
5	$\frac{15}{8}$	$\frac{10}{3}$
6	$\frac{5}{32}$	$\frac{8}{5}$
7	$\frac{35}{16}$	$\frac{28}{45}$
8	$\frac{35}{256}$	$\frac{64}{315}$
9	$\frac{315}{128}$	$\frac{2}{35}$
10	$\frac{63}{512}$	$\frac{8}{567}$
11	$\frac{693}{256}$	$\frac{44}{14175}$
12	$\frac{231}{2048}$	$\frac{32}{51975}$
13	$\frac{3003}{1024}$	$\frac{52}{467775}$
14	$\frac{429}{4096}$	$\frac{16}{868725}$
15	$\frac{6435}{2048}$	$\frac{8}{2837835}$
16	$\frac{6435}{65536}$	$\frac{256}{638512875}$
17	$\frac{109395}{32768}$	$\frac{34}{638512875}$
18	$\frac{12155}{131072}$	$\frac{8}{1206079875}$
19	$\frac{230945}{65536}$	$\frac{76}{97692469875}$
20	$\frac{46189}{524288}$	$\frac{32}{371231385525}$