A247335 and A247512 (also featuring A078986, A078987, A246643 and A246645).

On a geometric application suggested by Kival Ngaokrajang.
Wolfdieter Lang, Sep 29 2014.

See A240926 and A115032 for a similar geometric interpretation by Kival Ngaokrajang with illustrations.

See the W. Lang link ``Proof of the coincidence of a(n) with the touching circle problem (part II)." under A240926 where the general formulas are found in part I for touching circle problem in the larger part of the bisection, and in part II) for the smaller part of the bisection.

I) A247335: The touching circle sequence in the larger part of a circular disk with radius 10/9 (in some length units) bisected by a chord of length 4/3.

Here one has to adapt the values for h and R in part I) of the above mentioned link.
One starts with radius R=1 for the large circle C and h = 1/5 (the smaller sagitta for a chord of length 6/5). The circle C(n) with radius R(n) touches circle C, C(n-1) and the chord, with the input circle C(0) centered at [0, -9/10] with radius R(0) = 9/10 (the Cartesian coordinate system has origin O in the midpoint of the chord and the x-axis along the chord to the right). The center of C(n) is at [-x(n), -R(n)]. The procedure to find the recurrence for R(n) or the curvature 1/R(n) has been explained in the mentioned link. For the present case one finds for the relevant solution of the quadratic equation for R(n)

R(n) = R(n;-) = (-10*R(n-1)+ 19 - 2*sqrt(-100*R(n-1) + 90))*R(n-1)/(10*R(n-1) + 1)^2, with input R(0) = 9/10.

This produces the sequence [9/10, 9/100, 9/3610, 9/136900, 9/5198410, 9/197402500, ...], suggesting to rescale r(n) = (10/9)*R(n). The recurrence for r(n) is then

r(n) = (-9*r(n-1) + 19 - 6*sqrt(10)*sqrt(1-r(n-1)))*r(n-1)/(9*r(n-1) + 1)^2,
with input r(0) = 1.

The curvature b(n) = 1/r(n) satisfy then

b(n) = ((9 + b(n-1))^2)/(-9 + 19*b(n-1) - 6*sqrt(b(n-1)*(b(n-1) - 1))*sqrt(10))
with input b(0) =1.

This simplifies, after multiplying numerator and denominator by -9 + 19*b(n-1) + 6*sqrt(b(n-1)*(b(n-1) - 1))*sqrt(10) to

b(n) = -9 + 19*b(n-1) + 60*sqrt(b(n-1)*(b(n-1) - 1)/10)
with input b(0) = 1.

This is the sequence [1, 10, 361, 13690, 519841, 19740250, 749609641, 28465426090, 1080936581761, ...] = A247335, found by Kival Ngaokrajang.

The unique solution of this recurrence with input proceeds like explained in the mentioned link, part I).
Consider Y(n) := sqrt(b(n)*(b(n) - 1)/10), i.e., b(n) = -9 + 19*b(n-1) + 60*Y(n-1).

This is the sequence [0, 3, 114, 4329, 164388, 6242415, 237047382, 9001558101,...].

The first equation can be solved for b(n), yielding (the positive solution is relevant)

b(n) = (1 + sqrt(1 + 10*(2*Y(n))^2)/2, n >= 0, which implies

(2*b(n) - 1)^2 = 1 + 10*(2*Y(n))^2

The well known Pell equation x^2 - 10*y^2 = +1 has all the positive integer solutions given by (A078986(n), 6*A078987(n-1)), n>=0, starting with the pairs (1,0), (19,6), (721,228), (27379,8658), ... (see the on-line program mentioned in the link referred to earlier).
They are expressed in terms off Chebyshev polynomials S(n, x=38) (see A049310), namely A078986(n) = T(n, 19) = (S(n, 38) - S(n-2,38))/2 and A078987(n-1) = S(n-1, 38).
Therefore one has found integer solutions for the original recurrence, with Y(n) = 3*S(n-1, 38) and

b(n) = (1 + A078986(n))/2 = (2 + S(n, 38) - S(n-2, 38))/4, for n >=0.

This is indeed the sequence A247335(n) found by Kival Ngaokrajang.

II) A247512 (with A246643 and A246645): The touching circle sequence in the smaller part of a circular disk with radius 10/9 (in some length units) bisected by a chord of length 4/3.

See the above mentioned W. Lang link under A240926, part II).
The circle C'(n) of radius R'(n) touches the chord, the large input circle C (see part I) above) and C'(n-1), with the input circle C'(0) centered at [0, +1/10] with radius R'(0) = 1/10. The relevant solution of the quadratic equation becomes her, with R=1 and h = 1/5

R'(n) = R'(n;-) = (9*(-10*R'(n-1) + 11 - 2*sqrt(-100*R'(n-1)+10))*R'(n-1))/(10*R'(n-1)+9)^2 , with input R'(0) = 1/10.

This produces the sequence[[1/10, 9/100, 81/1210, 729/16900, 6561/259210], ...], suggesting to rescale r'(n) :=(10/9)*R'(n). This is then the situation considered by Kival Ngaokrajang with the large circle C' of radius 10/9 and the smaller sagitta of length 2/9. This leads to the following recurrence for the curvatures b'(n) = 1/r'(n).

b'(n) = (11*b'(n-1) - 9 + 20*sqrt((b'(n-1) - 9)*b'(n-1)/10))/9
with the input b'(0) = 9.

This is the sequence [9, 10, 121/9, 1690/81, 25921/729, 420250/6561, 7027801/59049, 119508490/531441, 2050368961/4782969, 35341836010/43046721, 610665665401/387420489,...].
The floor function produces Kival Ngaokrajang's sequence [9, 10, 13, 20, 35, 64, 119, 224, 428, 821, 1576 ,...] = A247512.
However, one can find the explicit form for the rational sequence {b'(n)} in terms of Chebyshev polynomials if one first redefine, this time n-dependent,

B'(n) = 9^(n-1)*b'(n),

which is the sequence [1, 10, 121, 1690, 25921, 420250, 7027801, 119508490, ...] = A246643. The one step recurrence is

B'(n) = 11*B'(n-1) - 9^(n-1) + 20*sqrt((B'(n-1) - 9^(n-1))*B'(n-1)/10)
with input B'(0) =1.

This sequence was not yet in OEIS and superseeker@oeis.org conjectured an o.g.f., which, after factorization of the denominator, looked like:

G(x) = (1 - 21*x + 90*x^2)/((1 - 9*x)*(1 - 22*x + 81*x^2)) ,

or in partial fraction decomposition

G(x) = (1/2)*((1 -11*x)/(1 - 22*x + 81*x^2) - 1/(1 - 9*x)).

Now, the o.g.f. 1/(1 - 22*x + 81*x^2) produces the sequence [1, 22, 403, 7084, 123205, 2136706, 37027927, 641541208, 11114644489, 192557340910,...] = A246645, which is 9^n*S(n, 22/9), with Chebyshev's S-polynomials (see A049310).
Thus, the superseeker's conjecture is

B'(n) = (9^n)*(1 + S(n, 22/9) - (11/9)*S(n-1, 22/9))/2,

or for the sequence of rational curvatures

b'(n) = B'(n)/9^(n-1) = (9/2)*(1 + S(n, 22/9) - (11/9)*S(n-1, 22/9)) , n >= 0.

For the proof that the b'(n) (or B'(n)) recurrence is indeed satisfied with this conjectured expression, consider

Y(n) := sqrt((B'(n) - 9^n)*B'(n)/10)), producing the sequence [0, 1, 22, 403, 7084, 123205, 2136706, 37027927, 641541208, 11114644489, ...] which looks like A246645(n-1) with A246645(-1) = 0.
After inserting the conjectured form for B'(n) in terms of S-polynomials into Y(n) the identity

S(n, 22/9)*S(n-1, 22/9) = (-1 + S(n, 22/9)^2 + S(n-1, 22/9)^2)/(22/9),

is used. This identity follows from the Cassini-Simson identity and the three term recurrence of the S-polynomials (see the mentioned W. Lang link, part IIIa) where these identities where used for x =3 instead of x = 22/9). This boils down to

Y(n)^2 = (B'(n) - 9^n)*B'(n)/10 = (9^(n-1)*S(n-1, 22/9))^2,

proving that Y(n) = A246645(n-1). The above given B'(n) recurrence becomes with this Y(n-1): B'(n) = 11*B'(n-1) - 9^(n-1) + 20*Y(n-1) = 11*(9^(n-1))*(1 + S(n-1, 22/9) - (11/9)*S(n-2, 22/9))/2 - 9^(n-1) + 20*9^(n-2)*S(n-2, 22/9) = (1/18)*9^n*(9 + 11*S(n-1, 22/9)- 9*S(n-2, 22/9) = (1/2)*9^(n-1)*(9- 11*S(n-1, 22/9)+ 9*S(n, 22/9) after the recurrence for 9*S(n-2, 22/9) = 22*S(n-1, 22/9) - 9*S(n, 22/9) has been employed. But this is indeed B'(n) = (1/2)*(9^n)*(1 + S(n, 22/9) - (11/9)*S(n-1, 22/9)) which ends the proof.

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