Notations

We use the definition of $T(n, k) = \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil$ for $1 \le n$ and $1 \le k \le \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right) \right\rfloor = \text{row}(n)$, and $S(n, k) = T(n, k) - T(n, k + 1)$ from A237591 and A237593, respectively.

Lemma

Let $i, n \in \mathbb{N}$ with $n \ge 3$ and $i \ge 1$. Equivalent are:

(1) $n = 2^{i-1} \times p$, where $1 < 2^i \le \text{row}(n)$, p is a prime and $p > \text{row}(n)$.

(2) (i) $T(n, 2^i) = T(n - 1, 2^i) + 1$,

(ii) for all $k \ne 2^i$, $1 < k \le \text{row}(n)$, $T(n, k) = T(n - 1, k)$,

(iii) $\sigma(n) = 2 \times n - 2 \times T(n, 2^i)$.

Theorem

The symmetric representation of $\sigma(n)$ consists of two regions of width 1 each with $2^k - 1$ upper/right boundary sides precisely when $n = 2^{k-1} \times p$ where $1 \le 2^k \le \text{row}(n) < p$ for some $k \in \mathbb{N}$ and prime p.

Corollary

The symmetric representation of $\sigma(n)$ consists of two regions of width 1 that meet at the diagonal, so that $\sigma(n) = 2 \times n - 2$, precisely when $n = 2^{(2^m - 1)} \times \left(2^{2^m} + 1\right)$ where $2^{2^m} + 1$ is a Fermat prime.

This sequence of numbers n is: 3, 10, 136, 32896, 2147516416, …[?]… (A191363).

Proof of Lemma "(1) ⇒ (2)"

(2.i) $T(n, 2^i) = \left\lceil \frac{2^{i-1} \times p + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = \frac{p-1}{2} - 2^{i-1} + \left\lceil \frac{1}{2^i} \right\rceil = \frac{p-1}{2} - 2^{i-1} + 1$

$T(n-1, 2^i) = \left\lceil \frac{2^{i-1} \times p - 1 + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = \frac{p-1}{2} - 2^{i-1}$.

(2.ii) For any $1 < k \le \text{row}(n)$, let $n = q \times k + d$ with $q, k \in \mathbb{N}$ and $0 \le d < k$.

If k is odd, simple calculations establish the equation.

If k is even and has an odd factor, then the three cases $0 < d < \frac{k}{2}$, $\frac{k}{2} < d < k$ and $d = \frac{k}{2}$ need to be considered.

Finally, if $k = 2^j$ with $j \ge 1$ and $j \ne i$, considering the two cases $1 \le j < i$ and $j > i$, the last with the three subcases $1 < d < 2^{j-1}$, $2^{j-1} < d < 2^j$ and $d = 2^{j-1}$ establish the equation.

(2.iii) $\sigma(n) = \sigma\left(2^{i-1} \times p\right) = \left(2^i - 1\right) \times (p + 1) = 2^i \times p + 2^i - p - 1$ and

$T(n, 2^i) = \left\lceil \frac{2^{i-1} \times p + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = \frac{p-1}{2} - 2^{i-1} + \left\lceil \frac{1}{2^i} \right\rceil = \frac{p-1}{2} - 2^{i-1} + 1$.

Proof of Lemma "(2) ⇒ (1)"

Suppose that $n = q \times 2^{i-1} + d$ with $q \in \mathbb{N}$ and $0 \le d < 2^{i-1}$. From assumption (2.i) we get:

$T(n, 2^i) = \left\lceil \frac{q \times 2^{i-1} + d + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d+1}{2^i} \right\rceil - 2^{i-1}$

$T(n-1,\ 2^i)\ =\ \left\lceil \frac{q\times 2^{i-1}+d\ -1+1}{2^i} - \frac{2^i+1}{2} \right\rceil\ =\ \left\lceil \frac{q-1}{2} + \frac{d}{2^i} \right\rceil - 2^{i-1}$

so that $\left\lceil \frac{q-1}{2} + \frac{d+1}{2^i} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d}{2^i} \right\rceil + 1$ must hold.

If q is odd, then $\left\lceil \frac{d+1}{2^i} \right\rceil = \left\lceil \frac{d}{2^i} \right\rceil + 1$, so that $d = 0$.

If q is even, then $\left\lceil \frac{-1}{2} + \frac{d+1}{2^i} \right\rceil = \left\lceil \frac{-1}{2} + \frac{d}{2^i} \right\rceil + 1$ requires $d = 2^{i-1}$, a contradiction.

Therefore, $n = q \times 2^{i-1}$ for some odd $q \in \mathbb{N}$.

If n has an odd divisor $k \le \text{row}(n)$ then assumption (2.ii) $T(n, k) = T(n - 1, k)$ implies $\left\lceil \frac{n+1}{k} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$ which is a contradiction. Therefore, any odd divisor of n is larger than $\text{row}(n)$.

From $T(n,\ 2^i)\ =\ T(q\times 2^{i-1},\ 2^i)\ =\ \left\lceil \frac{q\times 2^{i-1}+1}{2^i} - \frac{2^i+1}{2} \right\rceil\ =\ \frac{q-1}{2} - 2^{i-1} + 1$, assumption (2.iii) implies $\sigma(n)\ =\ 2\times n - 2\times T(n,\ 2^i)\ =\ q\times 2^i - 2\times\left(\frac{q-1}{2} - 2^{i-1} + 1\right)\ =\ q\times 2^i - q + 2^i - 1\ =\ (2^i - 1)\times(q+1)$. Since $\sigma(2^{i-1}\times q)\ =\ \sigma(2^{i-1})\times\sigma(q)\ =\ (2^i - 1)\times(q+1)$, $\sigma(q)\ =\ q + 1$ so that q must be a prime.

Proof of Theorem

Observe that for any $n \in \mathbb{N}$, $T(n, 1) = n$, $T(n, 2) = \left\lceil \frac{n}{2} \right\rceil - 1$ & $S(n, 1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$

Case 1: $k = 1$ and n is a prime.

Since n is odd, $S(n, 1) = S(n - 1, 1)$, i.e., the region is terminated with the first horizontal leg.
That leg has length $\frac{n+1}{2}$ and width 1.

Case 2: $k > 1$ and $n = 2^{k-1}\times p$

Since n is even, $S(n, 1) = S(n - 1, 1) + 1$, starting a region of width 1.
Since $T(n, i) = T(n - 1, i)$, for all $1 < i < 2^k$, the region continues with width 1
for $2^k - 1$ steps.

Proof of Corollary

Observe that $\text{row}\left(2^{(2^m-1)}\times\left(2^{2^m}+1\right)\right) = 2^{2^m}$ and $T\left(2^{(2^m-1)}\times\left(2^{2^m}+1\right),\ 2^{2^m}\right) = 1$.
The assertions now follow from the Theorem.