Notes on 2-periodic continued fractions and Lehmer sequences

Peter Bala, June 2014

Lehmer sequences

Let R and Q be nonzero integers such that $R - 4Q \neq 0$ and let α, β be the distinct roots of the quadratic equation $x^2 - \sqrt{R}x + Q = 0$. The Lehmer sequence $(L_n)_{n\geq 1}$ is an integer sequence defined by

$$L_n \equiv L_n(R,Q) = \begin{cases} (\alpha^n - \beta^n)\,/\,(\alpha - \beta) & \text{for } n \text{ odd} \\ \\ (\alpha^n - \beta^n)\,/\,\left(\alpha^2 - \beta^2\right) & \text{for } n \text{ even.} \end{cases} \tag{1}$$

Lehmer sequences were introduced in [1] as a generalization of Lucas sequences. Lehmer was mainly interested in the arithmetical properties of his sequences and restricted his attention to the case where both α and β were real, that is, he supposed both R and $R-4Q$ were positive integers. For our limited purposes in this note we don't assume these conditions hold.

The Lehmer sequence $L_n(R,Q)$ begins

$$[1, 1, R-Q, R-2Q, R^2 - 3RQ + Q^2, (R-Q)(R-3Q), ...]\,.$$

The sequence satisfies the pair of second-order linear recurrence equations:
$L_0 = 0, L_1 = 1$ and for $n \geq 1$

$$\begin{aligned} L_{2n} &= L_{2n-1} - QL_{2n-2} \\ \\ L_{2n+1} &= RL_{2n} - QL_{2n-1}. \end{aligned} \tag{2}$$

The ordinary generating function for the Lehmer sequence is easily calculated from (1) as

$$\sum_{n\geq 1} L_n x^n = \frac{x(1+x+Qx^2)}{1-(R-2Q)x^2+Q^2x^4} \tag{3}$$

$$= \frac{x+Qx^3}{1-(R-2Q)x^2+Q^2x^4} + \frac{x^2}{1-(R-2Q)x^2+Q^2x^4}\,. \tag{4}$$

as a sum of odd and even functions.
The Lehmer sequence $L_n(R,Q)$ may also be found by concatenating the first rows of the 2x2 matrices $\frac{1}{-Q}M^n$, $n = 1, 2, ...$, where

$$M = \begin{bmatrix} -Q & -Q \\ R & R-Q \end{bmatrix}.$$

We also note the product formula

$$L_n(R,Q) = \prod_{k=1}^{(n-1)/2} \left(R - 4Q\mathbf{cos}^2\left(\frac{k\pi}{n}\right)\right).$$

2-periodic continued fractions

Let P_1, P_2 and Q be nonzero integers and consider the 2-periodic generalized continued fraction

$$\frac{Q}{P_1+}\,\frac{Q}{P_2+}\,\frac{Q}{P_1+}\,\frac{Q}{P_2+}\cdots. \tag{5}$$

The sequence of convergents to the continued fraction begins

$$\frac{0}{1}, \frac{Q}{P_1}, \frac{QP_2}{P_1P_2+Q}, \frac{Q(P_1P_2+Q)}{P_1(P_1P_2+2Q)}, \frac{QP_2(P_1P_2+2Q)}{P_1^2P_2^2+3QP_1P_2+Q^2}, \frac{Q(P_1^2P_2^2+3QP_1P_2+Q^2)}{P_1(P_1P_2+Q)(P_1P_2+3Q)}, \dots. \tag{6}$$

Let $D_n = D_n(P_1,P_2,Q)$ (resp. $N_n = N_n(P_1,P_2,Q)$) denote the denominator (resp. the numerator) of the nth convergent of the continued fraction (5).
By considering the $(n+1)$th convergent of (5) we see that

$$\frac{N_{n+1}(P_1,P_2,Q)}{D_{n+1}(P_1,P_2,Q)} = \frac{Q}{P_1 + \frac{N_n(P_2,P_1,Q)}{D_n,(P_2,P_1,Q)}},$$

from which we find

$$N_{n+1}(P_1,P_2,Q) = QD_n(P_2,P_1,Q). \tag{7}$$

We will show that the sequence of denominators $(D_n)_{n\geq 1}$is essentially a Lehmer sequence. It will follow from this that the denominator sequence $(D_n)_{n\geq 1}$ is a linear divisibility sequence of the fourth order; furthermore, if both P_1 and P_2 are relatively prime to Q then we will show that the sequence $(D_n)_{n\geq 1}$ is a strong divisibility sequence, that is, $\gcd(D_n, D_m) = D_{\gcd(n,m)}$ for all natural numbers n and m. It follows from (7) that the same divisibility results also hold for the sequence of numerators $(N_{n+1})_{n\geq 1}$.

Recurrence equations and generating function for (D_n)

From the general theory of continued fractions the denominator sequence (D_n) satisfies the pair of second-order linear recurrence equations $D_0 = 0, D_1 = 1$ and for $n \geq 1$

$$\begin{aligned} D_{2n} &= P_1 D_{2n-1} + QD_{2n-2} \\ D_{2n+1} &= P_2 D_{2n} + QD_{2n-1}. \end{aligned} \tag{8}$$

We can use these recurrence equations to extend the sequence D_n to negative suffices. The result is

$$D_{-n} = (-1)^{n-1}\frac{D_n}{Q^n}. \tag{9}$$

The pair of recurrences (8) may be combined into the single fourth-order linear recurrence equation

$$D_n = (P_1P_2 + 2Q)D_{n-2} - Q^2 D_{n-4}, \quad n \geq 4. \tag{10}$$

Using (10), we can easily show the generating function for the sequence of denominators is

$$\begin{aligned} \sum_{n\geq 1} D_n x^n &= \frac{x(1 + P_1 x - Qx^2)}{1 - (P_1P_2 + 2Q)x^2 + Q^2x^4} && (11) \\ &= \frac{x - Qx^3}{1 - (P_1P_2 + 2Q)x^2 + x^4} + \frac{P_1 x^2}{1 - (P_1P_2 + 2Q)x^2 + x^4}\,. && (12) \end{aligned}$$

as a sum of odd and even functions. Comparing (12) with (4), we see that the sequence of denominators (D_n) of the convergents to the continued fraction (5) is essentially the Lehmer sequence $L_n(P_1P_2, -Q)$ with integer parameters P_1P_2 and $-Q$, except that the even-indexed denominators D_{2n} have an extra factor of P_1

$$D_n(P_1, P_2, Q) = \begin{cases} L_n(P_1P_2, -Q) & \text{for } n \text{ odd} \\ \\ P_1 L_n(P_1P_2, -Q) & \text{for } n \text{ even.} \end{cases} \tag{13}$$

By (7) there is the corresponding result for the sequence of numerators

$$N_{n+1}(P_1, P_2, Q) = \begin{cases} QL_n(P_1P_2, -Q) & \text{for } n \text{ odd} \\ \\ QP_2 L_n(P_1P_2, -Q) & \text{for } n \text{ even.} \end{cases} \tag{14}$$

In particular, if $P_1 = 1$ we have

$$D_n(1, P_2, Q) = L_n(P_2, -Q)\,,$$

while if $P_2 = 1$ we have

$$N_{n+1}(P_1, 1, Q) = QL_n(P_1, -Q)\,.$$

Divisibility properties of D_n

In [1] Lehmer states two theorems concerning the divisibility properties of Lehmer sequences. We sketch the proofs in the Appendix.

Theorem 1. The Lehmer sequence $L_n = L_n(R, Q)$ is a divisibility sequence, that is, L_n divides L_m whenever n divides m (provided $L_n \neq 0$). □

Corollary 1. Let P_1, P_2 and Q be nonzero integers. The sequence of denominators $(D_n(P_1, P_2, Q))_{n\geq 1}$ and the sequence of numerators$(N_{n+1}(P_1, P_2, Q))_{n\geq 1}$ of the convergents of the periodic continued fraction

$$\frac{Q}{P_1+}\,\frac{Q}{P_2+}\,\frac{Q}{P_1+}\,\frac{Q}{P_2+}\cdots$$

are divisibility sequences.

Proof. Immediate from (13) and (14). $\square$

Theorem 2. If R and Q are relatively prime integers then the Lehmer sequence $L_n = L_n(R, Q)$ is a strong divisibility sequence; that is, for all positive integers n, m we have

$$\gcd(L_n, L_m) = L_{\gcd(n,m)}. \square$$

Corollary 2. Let P_1, P_2 and Q be nonzero integers. Suppose further that both P_1 and P_2 are relatively prime to Q. Then the sequence of denominators $(D_n(P_1, P_2, Q))_{n\geq 1}$ and the sequence of numerators $(N_{n+1}(P_1, P_2, Q))_{n\geq 1}$ of the convergents of the periodic continued fraction

$$\frac{Q}{P_1+}\,\frac{Q}{P_2+}\,\frac{Q}{P_1+}\,\frac{Q}{P_2+}\cdots$$

are strong divisibility sequences.

Proof.

We prove the result for the sequence of denominators; the proof for the sequence of numerators will then follow from (7). The proof is on a case-by-case basis depending on the parity of n and m. For example, let us show that $\gcd(D_n, D_m) = D_{\gcd(n,m)}$ in the case when n is odd and m is even; the remaining cases are dealt with in a similar manner and are left for the reader.
First we prove the following property of the Lehmer sequence $L_n = L_n(P_1P_2, Q)$: P_1 is coprime to L_n when n odd, that is to say

$$\gcd(L_{2n-1}, P_1) = 1 \text{ for all positive integers } n. \tag{15}$$

The proof is by induction. We make the inductive hypothesis

$$\gcd(L_{2n-1}, P_1) = 1 \text{ for some } n.$$

This is clearly true when $n = 1$ since $L_1 = 1$. Using the recurrence equation for Lehmer sequences (2) we find

$$\begin{aligned} \gcd(L_{2n+1}, P_1) &= \gcd(P_1P_2L_{2n} - QL_{2n-1}, P_1) \\ &= \gcd(-QL_{2n-1}, P_1) \\ &= \gcd(L_{2n-1}, P_1) \quad \text{since by assumption } P_1 \text{ is relatively prime to } Q \\ &= 1 \end{aligned}$$

and the induction goes through.
Now since n is odd and m is even we have from (13)

$$\begin{aligned} \gcd(D_n, D_m) &= \gcd(L_n, P_1L_m) \\ &= \gcd(L_n, L_m) \quad \text{using (15)} \\ &= L_{\gcd(n,m)} \quad \text{by Theorem 2} \\ &= D_{\gcd(n,m)} \end{aligned}$$

using (13) again, since $\gcd(n, m)$ is odd. □

Appendix

In this section we sketch proofs of Lehmer's Theorem 1 and Theorem 2. We follow the treatment of the corresponding results for Lucas sequences given by Norfleet [3, Theorem 3]. It will be convenient to define the Lehmer sequence L_n by means of the recurrence equations (2), which we write in a condensed form as

$$L_n = f_n L_{n-1} - Q L_{n-2}, \quad [L_0 = 0, L_1 = 1], \tag{16}$$

where we define $f_j = 1$ if j is even and $f_j = R$ if j is odd.

Theorem 1. The Lehmer sequence $L_n = L_n(R, Q)$, with integer parameters R and Q, is a divisibility sequence.

Sketch proof. It is not difficult to establish inductively the following generalization of the defining recurrences (16):

for $n \geq 1$ and $k \geq 1$ there holds

$$L_{n+k} = \begin{cases} f_k L_{k+1} L_n - Q L_k L_{n-1} & \text{for } n \text{ even} \\ \\ L_{k+1} L_n - f_{k+1} Q L_k L_{n-1} & \text{for } n \text{ odd.} \end{cases} \tag{17}$$

Assuming this and choosing k to be a multiple of n in (17), say $k = mn$, gives an expression for $L_{(m+1)n}$ as a linear combination of L_n and L_{mn}. Therefore, if L_n divides L_{mn} then L_n divides $L_{(m+1)n}$. It follows by an induction argument that L_n is a divisibility sequence. □

Theorem 2. If R and Q are relatively prime integers then the Lehmer sequence $L_n = L_n(R, Q)$ is a strong divisibility sequence; that is, for all positive integers n, m we have

$$\gcd(L_n, L_m) = L_{\gcd(n,m)}\ .$$

Sketch proof. Firstl, one uses induction arguments to prove that for $n \geq 1$

$$\gcd(L_n, Q) = 1 \tag{18}$$

and

$$\gcd(L_{n+1}, L_n) = 1. \tag{19}$$

We need to prove the strong divisibility property

$$\gcd(L_n, L_m) = L_{\gcd(n,m)} \tag{20}$$

holds for all natural numbers n, m. We can assume without loss of generality that $n \geq m$. Let $k = n - m$. We begin by establishing the result

$$\gcd(L_n, L_m) = \gcd(L_{n-m}, L_m). \tag{21}$$

Suppose first that m is even. Then

$$\begin{aligned}
\gcd(L_n, L_m) &= \gcd(L_{m+k}, L_m) \\
&= \gcd(f_k L_{k+1} L_m - Q L_k L_{m-1}, L_m) \quad \text{by (17)} \\
&= \gcd(Q L_k L_{m-1}, L_m) \\
&= \gcd(L_k L_{m-1}, L_m) \quad \text{using (18)} \\
&= \gcd(L_k, L_m) \quad \text{using (19)} \\
&= \gcd(L_{(n-m)}, L_m).
\end{aligned}$$

The proof of (21) when m is odd is exactly similar.
We are now ready to prove the strong divisibility property (20) by means of a strong induction argument on $n + m$. Clearly, (20) is true for the base case $n = m = 1$. We make the inductive hypothesis that (20) is true for all n, m with $n + m \leq N$. Then if $n + m = N + 1$

$$\begin{aligned}
\gcd(L_n, L_m) &= \gcd(L_{n-m}, L_m) \text{ by (21)} \\
&= L_{\gcd(n\,-\,m,\,m)} \quad \text{by the inductive hypothesis} \\
&= L_{\gcd(n,\,m)}
\end{aligned}$$

and hence the induction goes through. □

If we examine the above proofs of Theorem 1 and Theorem 2 we see that they only use the fact that the ring of integers $\mathbb{Z}$ is an integral domain having a greatest common divisor function. Thus these two theorems may be generalized to Lehmer sequences defined by the recurrence equations (2) where now R and Q are taken to be elements of an arbitrary GCD domain. For example, the sequence of bivariate polynomials $P_n(x, y)$ defined by the linear recurrence equations $P_0 = 0, P_1 = 1$ and for $n \geq 1$

$$\begin{aligned}
P_{2n} &= P_{2n-1} - xy P_{2n-2} \\
P_{2n+1} &= (x + y)^2 P_{2n} - xy P_{2n-1}
\end{aligned} \tag{22}$$

will be a strong divisibility sequence in the polynomial ring $\mathbb{Z}[x, y]$. An explicit formula is

$$P_n(x, y) = \begin{cases} (x^n - y^n) \,/\, (x - y) & \text{for } n \text{ odd} \\ (x^n - y^n) \,/\, (x^2 - y^2) & \text{for } n \text{ even.} \end{cases} \tag{23}$$

LINKS

[1] D. H. Lehmer, An extended theory of Lucas' Functions, Annals of Mathematics Second Series, Vol. 31, No. 3 (July 1930), 419-448.
[2] M. Norfleet, Characterization of second-order strong divisibility sequences of polynomials, The Fibonacci Quarterly, Vol. 43, No. 2 (May 2005), 166-169.
[3] Wikipedia, Lehmer number
[4] Wikipedia, Lucas sequence