Maple-assisted proof of empirical formula for A241618

Robert Israel

Sep 3 2019

There are 91 possible last two elements for an array counted in $a(n)$. We list those as r_1, ..., r_{91} :

```
> r:= [seq(seq([i,j],j=0..12-i),i=0..12)]: nops(r);
```

91 **(1)**

Thus $a(n) = e^T v(n)$ where e is the vector of all 1's and $v_i(n)$ is the number of such arrays with last two elements r_i. We have $v(0) = e$ and $v(n+1) = T\,v(n)$ where T is the 91×91 matrix such that $T_{ij} = 1$ if the last element in r_j is the first element of r_i and the sum of these three elements is at most 12, and 0 otherwise.

```
> T:= Matrix(91,91,proc(i,j) if r[i][1]=r[j][2] and r[i][1]+r[i][2]
  +r[j][1]<=12 then 1 else 0 fi end proc):
```

To check, we compute the first few terms of the sequence. .

```
> U[0]:= Vector(91,1):
  for n from 1 to 20 do U[n]:= T . U[n-1] od:
  seq(U[0]^%T . U[j], j=1..20);
```

455, 3185, 22295, 145873, 980031, 6645821, 44678543, 300535053, 2025793471, 13644835113, 91879275469, 618858084619, 4168290681519, 28073432645895, 189079333842687, 1273493381875147, 8577194140275861, 57768891197339641, 389084375245995264, 2620554431767163871 **(2)**

Now the empirical formula is

```
> Emp:= a(n)=5*a(n-1)+4*a(n-2)+90*a(n-3)-178*a(n-4)-304*a(n-5)
  -2190*a(n-6)+3122*a(n-7)+4598*a(n-8)+27226*a(n-9)-35344*a(n-10)
  -38678*a(n-11)-204872*a(n-12)+263624*a(n-13)+217061*a(n-14)
  +1075988*a(n-15)-1372951*a(n-16)-878599*a(n-17)-4219367*a(n-18)
  +5250913*a(n-19)+2744168*a(n-20)+12887744*a(n-21)-15526366*a
  (n-22)-6882472*a(n-23)-31586623*a(n-24)+36749263*a(n-25)
  +14188650*a(n-26)+63480108*a(n-27)-71446109*a(n-28)-24250677*a
  (n-29)-106519271*a(n-30)+116299143*a(n-31)+34801486*a(n-32)
  +151401265*a(n-33)-160843414*a(n-34)-42327343*a(n-35)-184438893*a
  (n-36)+191181176*a(n-37)+44103326*a(n-38)+194263826*a(n-39)
  -196998632*a(n-40)-39713644*a(n-41)-178357453*a(n-42)+177255097*a
  (n-43)+31177560*a(n-44)+143508464*a(n-45)-140055290*a(n-46)
  -21470525*a(n-47)-101753388*a(n-48)+97670156*a(n-49)+13034856*a
  (n-50)+63756312*a(n-51)-60278612*a(n-52)-7010123*a(n-53)
  -35431750*a(n-54)+33079184*a(n-55)+3341113*a(n-56)+17464463*a
  (n-57)-16108575*a(n-58)-1412948*a(n-59)-7663035*a(n-60)+7014279*a
  (n-61)+531109*a(n-62)+2975276*a(n-63)-2697014*a(n-64)-174791*a
  (n-65)-1031090*a(n-66)+933657*a(n-67)+51574*a(n-68)+312146*a
  (n-69)-279926*a(n-70)-12762*a(n-71)-84813*a(n-72)+76728*a(n-73)
  +2915*a(n-74)+19414*a(n-75)-17311*a(n-76)-499*a(n-77)-4067*a
  (n-78)+3720*a(n-79)+86*a(n-80)+651*a(n-81)-579*a(n-82)-8*a(n-83)
  -102*a(n-84)+96*a(n-85)+a(n-86)+9*a(n-87)-8*a(n-88)-a(n-90)+a
```

```
(n-91):
```

This corresponds to $e^T P(T)\ e = 0$ where $P(x)$ is the following polynomial of degree 91:

```
> P:= x^91 - add(coeff(rhs(Emp),a(n-i))*x^(91-i),i=1..91);
```

$$P := x^{91} - 5x^{90} - 4x^{89} - 90x^{88} + 178x^{87} + 304x^{86} + 2190x^{85} - 3122x^{84} - 4598x^{83} - 27226x^{82} + 35344x^{81} + 38678x^{80} + 204872x^{79} - 263624x^{78} - 217061x^{77} - 1075988x^{76} + 1372951x^{75} + 878599x^{74} + 4219367x^{73} - 5250913x^{72} - 2744168x^{71} - 12887744x^{70} + 15526366x^{69} + 6882472x^{68} + 31586623x^{67} - 36749263x^{66} - 14188650x^{65} - 63480108x^{64} + 71446109x^{63} + 24250677x^{62} + 106519271x^{61} - 116299143x^{60} - 34801486x^{59} - 151401265x^{58} + 160843414x^{57} + 42327343x^{56} + 184438893x^{55} - 191181176x^{54} - 44103326x^{53} - 194263826x^{52} + 196998632x^{51} + 39713644x^{50} + 178357453x^{49} - 177255097x^{48} - 31177560x^{47} - 143508464x^{46} + 140055290x^{45} + 21470525x^{44} + 101753388x^{43} - 97670156x^{42} - 13034856x^{41} - 63756312x^{40} + 60278612x^{39} + 7010123x^{38} + 35431750x^{37} - 33079184x^{36} - 3341113x^{35} - 17464463x^{34} + 16108575x^{33} + 1412948x^{32} + 7663035x^{31} - 7014279x^{30} - 531109x^{29} - 2975276x^{28} + 2697014x^{27} + 174791x^{26} + 1031090x^{25} - 933657x^{24} - 51574x^{23} - 312146x^{22} + 279926x^{21} + 12762x^{20} + 84813x^{19} - 76728x^{18} - 2915x^{17} - 19414x^{16} + 17311x^{15} + 499x^{14} + 4067x^{13} - 3720x^{12} - 86x^{11} - 651x^{10} + 579x^{9} + 8x^{8} + 102x^{7} - 96x^{6} - x^{5} - 9x^{4} + 8x^{3} + x - 1 \quad (3)$$

Since T is 91×91, its characteristic polynomial has that degree. In fact, P is the characteristic polynomial of T, as we now verify.

```
> P - LinearAlgebra:-CharacteristicPolynomial(T,x);
```

$$0 \quad (4)$$

Thus the Cayley-Hamilton theorem states that $P(T) = 0$, and this completes the proof.

```
> factor(P);
```

$$(x^{10} - x^{9} - 2x^{8} - 2x^{7} + 3x^{6} + x^{5} + 2x^{4} - x^{3} - x + 1)\,(x^{45} - 3x^{44} - 3x^{43} - 51x^{42} + 3x^{41} - 18x^{40} + 403x^{39} - 36x^{38} + 265x^{37} - 1823x^{36} + 480x^{35} - 1072x^{34} + 4975x^{33} - 1276x^{32} + 2057x^{31} - 8496x^{30} + 1812x^{29} - 2374x^{28} + 10460x^{27} - 1678x^{26} + 1783x^{25} - 9606x^{24} + 1014x^{23} - 857x^{22} + 6897x^{21} - 619x^{20} + 304x^{19} - 3928x^{18} + 221x^{17} - 32x^{16} + 1714x^{15} - 76x^{14} + 3x^{13} - 651x^{12} + 17x^{11} + 6x^{10} + 166x^{9} - x^{8} - 44x^{6} + 5x^{3} - 1) \quad (5)$$