Maple-assisted proof of empirical formula for A241615

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There are 55 possible last two elements for an array counted in $a(n)$. We list those as r_1, ..., r_{55} :

```
> r:= [seq(seq([i,j],j=0..9-i),i=0..9)]: nops(r);
```

55 **(1)**

Thus $a(n) = e^T v(n)$ where e is the vector of all 1's and $v_i(n)$ is the number of such arrays with last two elements r_i. We have $v(0) = e$ and $v(n+1) = T\,v(n)$ where T is the 55×55 matrix such that $T_{ij} = 1$ if the last element in r_j is the first element of r_i and the sum of these three elements is at most 9, and 0 otherwise.

```
> T:= Matrix(55,55,proc(i,j) if r[i][1]=r[j][2] and r[i][1]+r[i][2]
  +r[j][1]<=9 then 1 else 0 fi end proc):
```

To check, we compute the first few terms of the sequence. .

```
> U[0]:= Vector(55,1):
  for n from 1 to 20 do U[n]:= T . U[n-1] od:
  seq(U[0]^%T . U[j], j=1..20);
```

220, 1210, 6655, 34243, 180829, 963886, 5093737, 26932543, 142701909, 755538278, 3999038946, 21172904049, 112098384491, 593455432350, 3141868198978, 16633824615067, 88062718713584, 466221475528171, 2468274573927916, 13067553701179851 **(2)**

Now the empirical formula is

```
> Emp:= a(n) = 4*a(n-1) +2*a(n-2) +44*a(n-3) -69*a(n-4) -79*a(n-5)
  -507*a(n-6) +572*a(n-7) +514*a(n-8) +2973*a(n-9) -3097*a(n-10)
  -1820*a(n-11) -10364*a(n-12) +10800*a(n-13) +4269*a(n-14) +25019*
  a(n-15) -25821*a(n-16) -6914*a(n-17) -44207*a(n-18) +44275*a
  (n-19) +8829*a(n-20) +59359*a(n-21) -57787*a(n-22) -9308*a(n-23)
  -62456*a(n-24) +58989*a(n-25) +8291*a(n-26) +52174*a(n-27)
  -48385*a(n-28) -5846*a(n-29) -35493*a(n-30) +32403*a(n-31) +3452*
  a(n-32) +19719*a(n-33) -17810*a(n-34) -1563*a(n-35) -9053*a(n-36)
  +8178*a(n-37) +608*a(n-38) +3390*a(n-39) -3025*a(n-40) -167*a
  (n-41) -1072*a(n-42) +973*a(n-43) +42*a(n-44) +259*a(n-45) -227*a
  (n-46) -6*a(n-47) -56*a(n-48) +52*a(n-49) +a(n-50) +7*a(n-51) -6*
  a(n-52) -a(n-54) +a(n-55):
```

This corresponds to $e^T P(T)\, e = 0$ where $P(x)$ is the following polynomial of degree 55:

```
> P:= x^55 - add(coeff(rhs(Emp),a(n-i))*x^(55-i),i=1..55);
```

$$P := x^{55} - 4x^{54} - 2x^{53} - 44x^{52} + 69x^{51} + 79x^{50} + 507x^{49} - 572x^{48} - 514x^{47} - 2973x^{46} + 3097x^{45} + 1820x^{44} + 10364x^{43} - 10800x^{42} - 4269x^{41} - 25019x^{40} + 25821x^{39} + 6914x^{38} + 44207x^{37} - 44275x^{36} - 8829x^{35} - 59359x^{34} + 57787x^{33} + 9308x^{32} + 62456x^{31} - 58989x^{30} - 8291x^{29} - 52174x^{28} + 48385x^{27} + 5846x^{26} + 35493x^{25} - 32403x^{24} - 3452x^{23} - 19719x^{22} + 17810x^{21} + 1563x^{20} + 9053x^{19} - 8178x^{18} \quad \textbf{(3)}$$

$$- 608\, x^{17} - 3390\, x^{16} + 3025\, x^{15} + 167\, x^{14} + 1072\, x^{13} - 973\, x^{12} - 42\, x^{11} - 259\, x^{10}$$
$$+ 227\, x^{9} + 6\, x^{8} + 56\, x^{7} - 52\, x^{6} - x^{5} - 7\, x^{4} + 6\, x^{3} + x - 1$$

Since T is 55×55, its characteristic polynomial has that degree. In fact, P is the characteristic polynomial of T, as we now verify.

```
> P - LinearAlgebra:-CharacteristicPolynomial(T,x);
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$$0 \qquad \textbf{(4)}$$

Thus the Cayley-Hamilton theorem states that $P(T) = 0$, and this completes the proof.