Numbers $n = 2^{k-1} \times p^{2h}$ where $2^k < p$ and $p^h \leq \text{row}(n) < 2^k \times p^h$ and $p \geq 3$ is a Prime are Precisely those whose Symmetric Representation of $\sigma(n)$ Consists of an Odd Number Regions of Width 1.

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2015-06-01

Notations

We use the definition of $T(n, i) = \left\lceil \frac{n+1}{i} - \frac{i+1}{2} \right\rceil$ for $1 \leq n$ and $1 \leq i \leq \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right) \right\rfloor = \text{row}(n)$, and $S(n, i) = T(n, i) - T(n, i + 1)$ from A237591 and A237593, respectively. Observe that $T(n, 1) = n$.

Lemma

Let $n, k, h, p \in \mathbb{N}$ where $k \geq 1, h \geq 0$ and $p \geq 3$ is prime satisfying $2^k < p$. Equivalent are:

(1) $n = 2^{k-1} \times p^{2h}$.

(2) (a) $p^h \leq \text{row}(n) < 2^k \times p^h$,

(b.i) $T(n, p^j) = T(n - 1, p^j) + 1$, for all $0 \leq j \leq h$,

(b.ii) $T(n, 2^k p^j) = T(n - 1, 2^k p^j) + 1$, for all $0 \leq j < h$,

(b.iii) $T(n, p^j) - T(n, 2^k p^j) = \frac{1}{2} \times (2^k - 1) \times (p^j + p^{2h-j})$, for all $0 \leq j < h$,

(c) $T(n, i) = T(n - 1, i)$, for all $1 < i \leq \text{row}(n)$, except for $i \neq p^j, 0 \leq j \leq h$, and $i \neq 2^k p^j, 0 \leq j < h$.

Theorem

The symmetric representation of $\sigma(n)$ consists of an odd number of regions of width one precisely when $n = 2^{k-1} \times p^{2h}$, where $n, k, h, p \in \mathbb{N}, k \geq 1, h \geq 0$ and $p \geq 3$ is a prime satisfying $2^k < p$.

In this case there are $2 \times h + 1$ regions in the symmetric representation of $\sigma(n)$ of respective sizes $\frac{1}{2} \times (2^k - 1) \times (p^j + p^{2h-j}), 0 \leq j \leq 2 \times h$. The first $j = 0, \ldots, h-1$ sections, symmetrically duplicated, start with the p^j-th leg in the Dyck path. The center section starts at leg p^h, extends symmetrically beyond the center of the path and has size $(2^k - 1) \times p^h$.

Proof of Lemma "(1) ⇒ (2)"

(2.a) By definition, the number of elements in the n-th row of T(n, _) is

$$\text{row}(2^{k-1} \times p^{2h}) = \left\lfloor \frac{1}{2}\left(\sqrt{2^{k+2} \times p^{2h} + 1} - 1\right) \right\rfloor.$$

For the first inequality reduces to: $p^h + 1 \leq 2^k \times p^h$
and the second to: $p^h < 2^k \times p^h + 1$
which hold since $h \geq 0, k \geq 1$ and $p \geq 3$.

(2.b) Since $2^k < p$ and $p^h \leq \text{row}(n)$, all terms in section (2.b) are well-defined and direct evaluations establish the three claimed identities.

(2.c) For any $1 < i \le \text{row}(n)$, let $n = q \times i + d$ with $q, i, d \in \mathbb{N}$ and $0 \le d < i$.
Then $T(n, i) - T(n-1, i) = \left\lceil \frac{d+1}{i} - \frac{i+1}{2} \right\rceil - \left\lceil \frac{d}{i} - \frac{i+1}{2} \right\rceil$.

Let i be odd and assume that $d = 0$. Then $i|n$ so that $i = p^j$, for $0 \le j \le h$, but those values are excluded.
Therefore, $d > 0$ and $T(n, i) - T(n-1, i) = \left\lceil \frac{d+1}{i} \right\rceil - \left\lceil \frac{d}{i} \right\rceil = 1 - 1 = 0$.

Let i be even, say $i = 2 \times s$. Then
$T(n, i) - T(n-1, i) = \left\lceil \frac{d+1}{2s} - \frac{2s+1}{2} \right\rceil - \left\lceil \frac{d}{2s} - \frac{2s+1}{2} \right\rceil = \left\lceil \frac{d+1}{2s} - \frac{1}{2} \right\rceil - \left\lceil \frac{d}{2s} - \frac{1}{2} \right\rceil$.
Case 1: $d < s$: Then each of the two terms equals zero.
Case 2: $d > s$: Then each of the two terms equals one.
Case 3: $d = s$: Then $n = 2 \times q \times s + s = (2 \times q + 1) \times s$ so that $2^{k-1}|s$. Therefore, $i = 2^k \times p^j$, for some $0 \le j < h$, which is excluded, so that this case does not occur.

Proof of Lemma "(2) ⇒ (1)"

First, observe that assumptions $2^k < p$ and (2.a) insure that $T(n, p^j)$, for all $0 \le j \le h$, and $T(n, 2^k p^j)$, for all $0 \le j < h$, are well defined.

Suppose that $n = q \times 2^{k-1} \times p^j + d$ with $q, d, j \in \mathbb{N}$, $0 \le j < h$, $k \ge 1$, and $0 \le d < 2^{k-1} \times p^j$.
From assumption (2.b.ii) we get:
$T(n, 2^k p^j) = \left\lceil \frac{q \times 2^{k-1} \times p^j + d + 1}{2^k \times p^j} - \frac{2^k \times p^j + 1}{2} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d+1}{2^k p^j} \right\rceil - 2^{k-1} p^j$
$T(n-1, 2^k p^j) = \left\lceil \frac{q \times 2^{k-1} \times p^j + d - 1 + 1}{2^k \times p^j} - \frac{2^k \times p^j + 1}{2} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d}{2^k \times p^j} \right\rceil - 2^{k-1} \times p^j$
so that $\left\lceil \frac{q-1}{2} + \frac{d+1}{2^k \times p^j} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d}{2^k \times p^j} \right\rceil + 1$.
If q is odd, then $\left\lceil \frac{d+1}{2^k \times p^j} \right\rceil = \left\lceil \frac{d}{2^k \times p^j} \right\rceil + 1$, so that $d = 0$.
If q is even, then $\frac{q}{2} + \left\lceil \frac{-1}{2} + \frac{d+1}{2^k \times p^j} \right\rceil = \frac{q}{2} + \left\lceil \frac{-1}{2} + \frac{d}{2^k \times p^j} \right\rceil + 1$ requires $d = 2^k \times p^j$, a contradiction.
Therefore, $2^{k-1} \times p^j$, $0 \le j < h$, is a divisor of n.
If 2^k divides n, say $n = z \times 2^k$ for some $z \in \mathbb{N}$, then
$T(n, 2^k) = \left\lceil \frac{z \times 2^k + 1}{2^k} - \frac{2^k + 1}{2} \right\rceil = z - 2^{k-1} + \left\lceil \frac{-1}{2} + \frac{1}{2^k} \right\rceil = z - 2^{k-1}$
$T(n-1, 2^k) = \left\lceil \frac{z \times 2^k - 1 + 1}{2^k} - \frac{2^k + 1}{2} \right\rceil = z - 2^{k-1} + \left\lceil \frac{-1}{2} \right\rceil = z - 2^{k-1}$
which contradicts (2.b.ii) so that 2^{k-1} is the largest power of two dividing n.
Similarly, let $n = a \times p^h + b$ with $a, b \in \mathbb{N}$ and $0 \le b < p^h$ for prime $p \ge 3$. Then the expressions
$T(n, p^h) = \left\lceil \frac{a \times p^h + b + 1}{p^h} - \frac{p^h + 1}{2} \right\rceil = a - \frac{p^h + 1}{2} + \left\lceil \frac{b+1}{p^h} \right\rceil$
$T(n-1, p^h) = \left\lceil \frac{a \times p^h + b - 1 + 1}{p^h} - \frac{p^h + 1}{2} \right\rceil = a - \frac{p^h + 1}{2} + \left\lceil \frac{b}{p^h} \right\rceil$
satisfy $\left\lceil \frac{b+1}{p^h} \right\rceil = \left\lceil \frac{b}{p^h} \right\rceil + 1$ by (2.b.i), so that $b = 0$.
Therefore, p^h is a divisor of n. If n has an odd prime divisor $k \le \text{row}(n)$ with $k \ne p$ then $T(n, k) = T(n-1, k)$ holds by assumption (2.c). This, in turn, implies $\left\lceil \frac{n+1}{k} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$ which is a contradiction.
Therefore, p is the only odd prime divisor less than row(n).
Finally, suppose that $n = s \times 2^{k-1} \times p^h$ with $1 \le s \in \mathbb{N}$. Note that s must be odd. Then we get:
$T(n, 1) - T(n, 2^k) = s \times 2^{k-1} \times p^h - \left\lceil \frac{s \times 2^{k-1} \times p^h + 1}{2^k} - \frac{2^k + 1}{2} \right\rceil$
$= s \times 2^{k-1} \times p^h - \frac{s \times p^h - 1}{2} + 2^{k-1} - \left\lceil \frac{1}{2^k} \right\rceil = \frac{1}{2} \times (s \times 2^k \times p^h - s \times p^h + 1 + 2^k - 2)$
$= \frac{1}{2} \times ((2^k - 1) \times s \times p^h + (2^k - 1)) = \frac{1}{2} \times (2^k - 1) \times (s \times p^h + 1)$.

Now condition (2.b.iii) with $j = 0$ leads to equation:
$\frac{1}{2} \times (2^k - 1) \times (s \times p^h + 1) = \frac{1}{2} \times (2^k - 1) \times (p^{2h} + 1)$.
In other words, $s = p^h$, and $n = 2^{k-1} \times p^{2h}$.

Proof of Theorem

The lengths of the segments in the symmetric Dyck paths that bound the first half of the symmetric representation of $\sigma(n)$ are given by:

S(n, k) = T(n, k) - T(n, k + 1) for $1 \le n$ and $1 \le k \le$ row(n).

The four conditions (2.a), (2.b.i), (2.b.ii) & (2.c) together with T(n, 1) = n imply that the first h regions of $\sigma(n)$ extend from T(n, p^j) through T(n, $2^k p^j$) and have width 1, for all $0 \le j < h$, and that the region starting at leg p^h extends beyond the center of the Dyck path. The formula in (2.b.iii) establishes the size of each of the h symmetrically duplicated regions. so that with $\sigma(n) = (2^k - 1) \sum_{j=0}^{2h} p^j$, the size of the central region equals $(2^k - 1) \times p^h$.