Radiative Transfer Equations

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Modeling the passage of light through an absorbing and scattering medium (for example, a planetary atmosphere) is a difficult challenge. Its solution is applicable to neutron diffusion in nuclear reactor theory. We can hope only to present a few important integral equations and associated constants [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

0.1. Schwarzschild-Milne. Let $s \geq 0$. For a homogeneous semi-infinite plane-parallel atmosphere with isotropic scattering, the Milne equation [13, 14]

$$f(s) = \frac{\omega}{2} \int_0^\infty f(t) E_1(|s-t|)\, dt, \qquad f(0) = 1$$

arises, where $0 < \omega \leq 1$ is a constant (albedo) and

$$E_n(x) = \int_1^\infty \frac{e^{-x\,y}}{y^n} dy$$

for $n \geq 1$, which is $-\operatorname{Ei}(-x)$ if $n = 1$. Define

$$Z(\mu) = (1 - \omega\,\mu \operatorname{arctanh}(\mu))^2 + \frac{1}{4}\pi^2\omega^2\mu^2$$

and $H(\mu)$ exactly as later; we suppress the dependence on ω. In the special case when $\omega = 1$ (conservative case), the solution is given by [2]

$$f(s) = \sqrt{3}\,(s + q(s))$$

where

$$q(s) = \frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{3}} \int_0^1 \frac{1 - e^{-s/\mu}}{H(\mu)Z(\mu)} d\mu = q_\infty - \frac{1}{2\sqrt{3}} \int_0^1 \frac{e^{-s/\mu}}{H(\mu)Z(\mu)} d\mu$$

and q_∞ is **Hopf's constant** [1, 6, 10, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]:

$$\begin{aligned} q_\infty &= \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{3}{\sin(\theta)^2}-\frac{1}{1-\theta\cot(\theta)}\right)d\theta \\ &= \frac{6}{\pi^2}+\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{3}{\theta^2}-\frac{1}{1-\theta\cot(\theta)}\right)d\theta \\ &= \frac{6}{\pi^2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{b_{n+1}}{2n-1}\left(\frac{\pi}{2}\right)^{2n-1}=0.7104460895.... \end{aligned}$$

The series coefficients b_2, b_3, b_4, ... are defined recursively via

$$\sum_{k=1}^{n} a_k b_{n-k+1} = 0, \qquad b_1 = 3$$

where

$$a_k = \frac{(-1)^{k-1}2^{2k}B_{2k}}{(2k)!}$$

and $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, ... are the Bernoulli numbers.

Mark [25] was the first to solve Milne's equation, building on work by Wiener & Hopf [26] and Placzek & Seidel [15]. An integral equation for $q(s)$ directly is [27, 28]

$$q(s) = \frac{1}{2}E_3(s)+\frac{1}{2}\int_0^{\infty} q(t)E_1(|s-t|)\,dt$$

and a related formula for Hopf's constant is

$$q_\infty = \frac{3}{8}+\frac{3}{2}\int_0^{\infty} q(t)E_3(t)\,dt.$$

0.2. Ambarzumian-Chandrasekhar. Let $0 \le \mu \le 1$ and $0 < \omega \le 1$. The equation [4, 5, 29, 30, 31, 32, 33]

$$H(\mu) = 1+\frac{1}{2}\omega\,\mu\int_0^1 \frac{H(\mu)H(\lambda)}{\mu+\lambda}d\lambda$$

possesses a continuous solution; further, it is unique if $\omega = 1$. A better definition of $H(\mu)$ for arbitrary ω avoids ambiguity [2, 34]:

$$H(\mu) = f(0,\mu)$$

where

$$f(s,\mu)=e^{-s/\mu}+\frac{\omega}{2}\int\limits_0^{\infty}f(t,\mu)E_1\left(|s-t|\right)dt.$$

Halpern, Lueneburg & Clark [35] and Fock [36] proved that [10, 37]

$$\begin{aligned}H(\mu) &= \exp\left[-\frac{\mu}{\pi}\int\limits_0^{\infty}\ln\left(1-\omega\frac{\arctan(\lambda)}{\lambda}\right)\frac{d\lambda}{1+\mu^2\lambda^2}\right]\\ &= \exp\left[-\frac{\mu}{\pi}\int\limits_0^{\pi/2}\frac{\ln\left(1-\omega\,\theta\cot(\theta)\right)}{\cos(\theta)^2+\mu^2\sin(\theta)^2}d\theta\right]\end{aligned}$$

and it is clear that $H(0)=1$ and H increases with μ. Define moments

$$\alpha_n=\int\limits_0^1 H(\mu)\mu^n d\mu$$

then for $\omega=1$ we have [11, 38]

$$\alpha_0=2,\qquad \alpha_1=2/\sqrt{3}=1.1547005383...,$$

$$\alpha_2=\frac{2}{\sqrt{3}}q_\infty=\frac{2}{\sqrt{3}}\left(1-\eta_0\right)=0.8203524821...,$$

$$\alpha_3=\left(\frac{1}{5}+\frac{1}{3}q_\infty^2\right)\sqrt{3}=0.6378182680...,$$

$$\alpha_4=\frac{2}{\sqrt{3}}\left(\frac{1}{3}-\eta_2+\frac{3}{10}q_\infty+\frac{1}{6}q_\infty^3\right)=0.5222273037...$$

where

$$\eta_j=\int\limits_0^1\xi^j\left[\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{2\left(1-\xi\,\mathrm{arctanh}(\xi)\right)}{\pi\,\xi}\right)\right]d\xi$$

for $j\geq 0$. See also [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55].

Let us examine a generalization to finite atmospheres. Let $\tau>0$. The coupled equations

$$X(\mu)=1+\frac{1}{2}\omega\,\mu\int\limits_0^1\frac{X(\mu)X(\lambda)-Y(\mu)Y(\lambda)}{\mu+\lambda}d\lambda,$$

$$Y(\mu)=e^{-\tau/\mu}+\frac{1}{2}\omega\,\mu\int\limits_0^1\frac{Y(\mu)X(\lambda)-X(\mu)Y(\lambda)}{\mu-\lambda}d\lambda$$

give solutions related by

$$Y(\mu) = e^{-\tau/\mu} X(-\mu), \qquad X(\mu) = e^{-\tau/\mu} Y(-\mu)$$

(appropriately extended for $\mu < 0$), but the solutions are non-unique if $\omega = 1$. A better definition is

$$X(\mu) = f(0, \mu), \qquad Y(\mu) = f(\tau, \mu)$$

where

$$f(s, \mu) = e^{-s/\mu} + \frac{\omega}{2} \int_0^\tau f(t, \mu) E_1 (|s - t|) \, dt.$$

The only difference with before is that the upper limit of integration here is $\tau < \infty$; in fact,

$$\lim_{\tau \to \infty} X(\mu) = H(\mu), \qquad \lim_{\tau \to \infty} Y(\mu) = 0.$$

Integral expressions for X, Y analogous to H are not known. Clearly $X(0) = 1$, $Y(0) = 0$ and both X, Y increase with μ. Define moments

$$\alpha_n = \int_0^1 X(\mu) \mu^n d\mu, \qquad \beta_n = \int_0^1 Y(\mu) \mu^n d\mu$$

then for $\omega = 1$ the following hold [11]:

$$\alpha_0 + \beta_0 = 2, \qquad \alpha_1 - \beta_1 = \tau \, \beta_0, \qquad \alpha_2 + \beta_2 = \frac{2}{3\beta_0} - \frac{\tau}{2} (\alpha_1 + \beta_1)$$

for any τ. When $\tau = 1/10$, we have [24]

$$\alpha_0 = 1.1420220619..., \quad \alpha_1 = 0.5765390018..., \quad \alpha_2 = 0.3851978742...,$$

$$\beta_0 = 0.8579779380..., \quad \beta_1 = 0.4907412080..., \quad \beta_2 = 0.3384588719...$$

and when $\tau = 5$, we have

$$\alpha_0 = 1.8201574310..., \quad \alpha_1 = 1.0269371382..., \quad \alpha_2 = 0.7210212649...,$$

$$\beta_0 = 0.1798425689..., \quad \beta_1 = 0.1277242933..., \quad \beta_2 = 0.0992710166....$$

No exact formulas for α_n or β_n are known for $\tau < \infty$. The solutions $X(\mu)$, $Y(\mu)$ for the conservative case are not the same as the "standard solutions"

$$\tilde{X}(\mu) = X(\mu) + \frac{\beta_0 \mu}{\alpha_1 + \beta_1} (X(\mu) + Y(\mu)), \qquad \tilde{Y}(\mu) = Y(\mu) - \frac{\beta_0 \mu}{\alpha_1 + \beta_1} (X(\mu) + Y(\mu))$$

described by Chandrasekhar [56, 57], which satisfy $\tilde{\alpha}_0 = 2$ and $\tilde{\beta}_0 = 0$ (rather than the non-homogenous Milne equation for f). It is known that

$$\tilde{\alpha}_1^2 - \tilde{\beta}_1^2 = \frac{4}{3}$$

and, further, that pairwise moment sums are invariant [24]:

$$\alpha_1 + \beta_1 = 1.0672802099... = \tilde{\alpha}_1 + \tilde{\beta}_1,$$

$$\alpha_2 + \beta_2 = 0.7236567462... = \tilde{\alpha}_2 + \tilde{\beta}_2$$

when $\tau = 1/10$, and

$$\alpha_1 + \beta_1 = 1.1546614315... = \tilde{\alpha}_1 + \tilde{\beta}_1,$$

$$\alpha_2 + \beta_2 = 0.8202922816... = \tilde{\alpha}_2 + \tilde{\beta}_2$$

when $\tau = 5$. Conceivably an integral expression might exist for $\alpha_n + \beta_n$ but not for either α_n or β_n. Note that, in general [2],

$$\lim_{\mu \to \infty} H(\mu) = (1 - \omega)^{-1/2}$$

whereas

$$\lim_{\mu \to \infty} X(\mu) = \left[1 - \frac{\omega}{2}(\alpha_0 - \beta_0)\right]^{-1} = \lim_{\mu \to \infty} Y(\mu)$$

assuming $\tau < \infty$. See also [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72]. Much territory remains for exploration.

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