International Championship of Mathematical and Logical Games
Championnat International des Jeux Mathématiques & Logiques

http://homepage.hispeed.ch/FSJM/

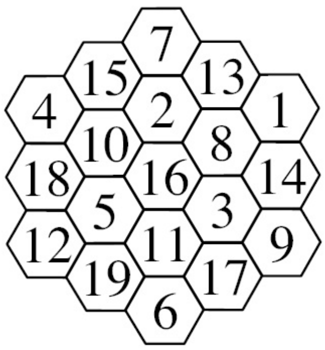
Individual Quarter Final 2008
Quart de finale individuels 2008

Problem 18 – Abella's Honeycomb

Abella is investigating honeycombs comprising identical regular hexagonal cells and forming themselves a regular hexagon. The order of the honeycomb corresponds to the number of cells on one side. The cells are numbered from 1 to the total number of cells within the honeycomb.

As for the honeycombs of order 3 (see the attached example), Abella realizes that it is possible to arrange the cell numbers in such a way that the difference of any two adjacent cell numbers is always at least 5, but for 6 this is impossible.

What is the greatest possible minimal difference of two adjacent cells in a honeycomb of order 5?[1]

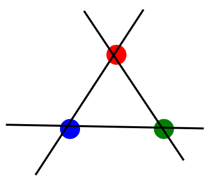


Procedure

1. Let's develop a strategy to fill in the numbers into a honeycomb of any order.
2. After that let's show that the given example corresponds to this strategy.
3. Then let's fill in a honeycomb of order 5.
4. And last, let's calculate the requested minimal difference thereof.

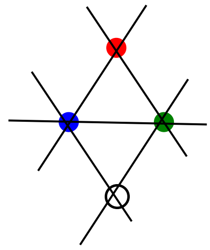
At the end we will have a formula for the greatest minimal difference *a(n)* for a honeycomb of any order *n*.

This problem deals with numbers arranged in triangles rather than in hexagons. The difference of the three numbers in the positions of the red, blue and green dots should never be less than a minimal difference:



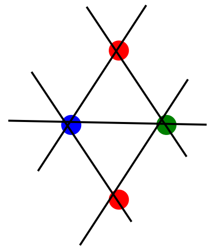
If we add one more dot (e. g. beneath, marked **O**), it has to have at least this difference from the blue and green dots, too.

[1] problem translated from homepage.hispeed.ch/FSJM/documents/22_Quarts_ind.pdf by

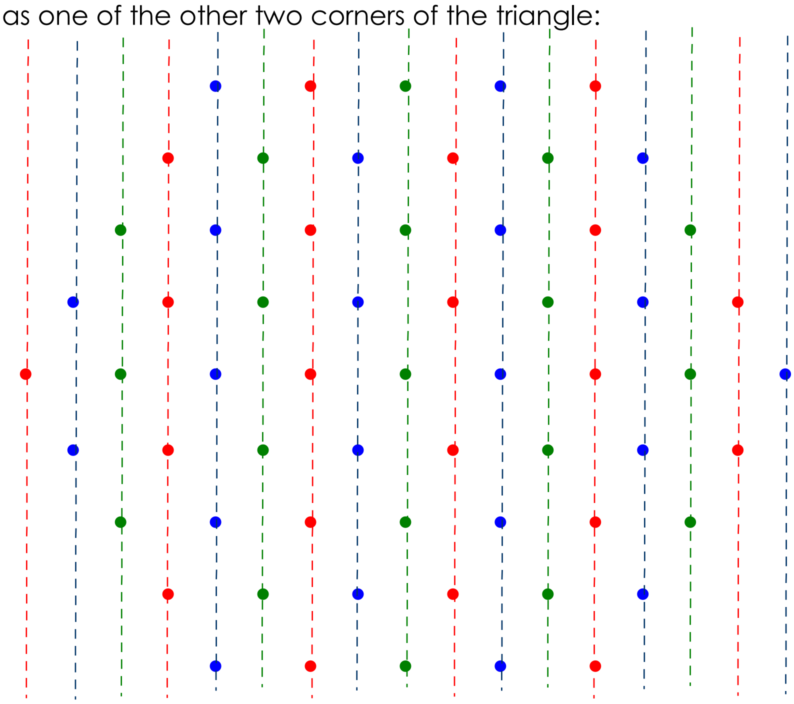


The difference from the red dot is irrelevant.

We are free to color it red, too:



Let's go on with all triangles of the honeycomb of order 5. Every additional dot must not be the same color as one of the other two corners of the triangle:



(The colored dashed lines do not represent the triangles, but merely link the dots of the same color.)

Thus we have 21 red, 21 blue and 19 green dots, 61 dots altogether. Let's divide the numbers from 1 to 61 into three groups.

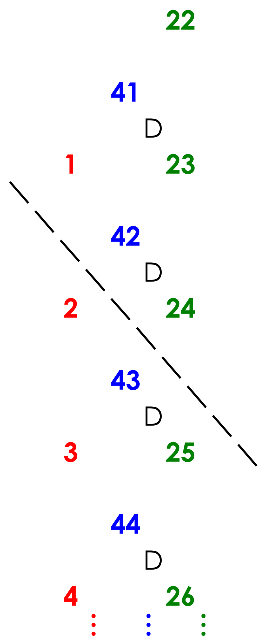
In order to maximize the mean difference between numbers of different groups we put e. g. the 21 smallest numbers into the red, the 21 biggest numbers into the blue, and the other 19 into the green group:

red	**1…21**
green	**22…40**
blue	**41…61**

Why creating three groups only and not more? The more groups we have, the smaller every group and the smaller the mean distance between the individual groups' numbers!

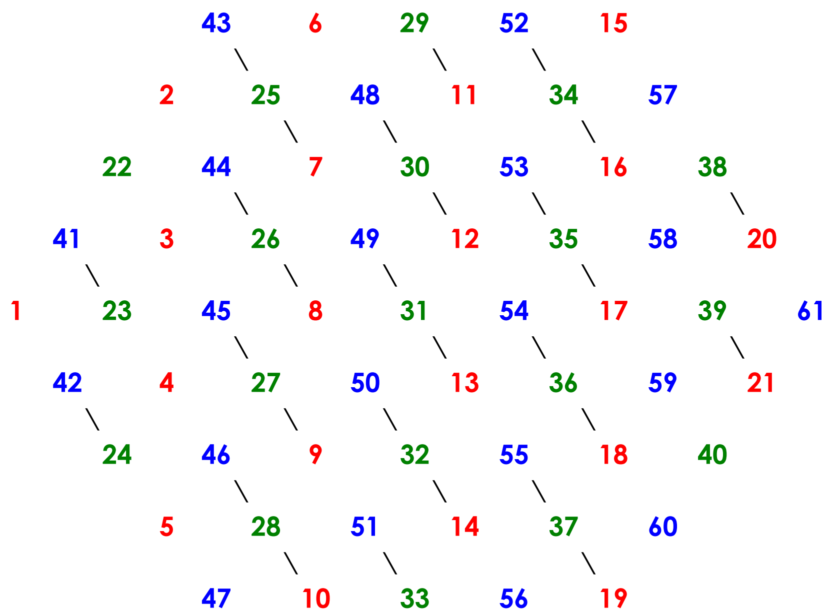
Furthermore it is advisable to keep the smaller numbers from all groups together as a whole as well as the corresponding bigger numbers. Say, from left to right and from top to bottom.

So we get – arranged in a linear, parallel mode (but disregarding the honeycomb structure as such yet) – the following picture:



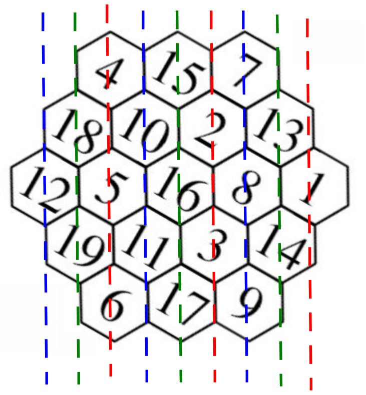
Where the smallest difference (marked *D*) is 18. In the honeycomb itself the vertical columns will be split (e. g. along the dashed line in the above picture) and probably shifted against each other, producing a smaller *D* (decreased by 1). This shifting and decreasing by 1 will occur every time we increase the factor *k* in the order *n* by 1, when $n = 3 \cdot k + 2$.

Let's fill in and check all of the 3! = 6 permutations of the three groups for its minimal *D*.



The above variant produces a minimal difference of **18** (marked by \). Some others only produce a minimum of 17 or less.

Rotating slightly the initially presented honeycomb, we realize exactly the same pattern (left and right side mirrored; minimal difference of 5; group sizes of 6, 7 and 6, totalling 19 cells):



Now let's deal with formulas.

The total number of cells in a honeycomb of order *n* is *s(n)*,

$$s(n) = 1 + \sum_{i=1}^{n-1} 6 \cdot i = 1 + 6 \cdot \binom{n}{2} = 3 \cdot n \cdot (n-1) + 1$$

The smallest group contains not more than $\frac{s(n)}{3}$ cells, which is also the greatest possible minimal difference Δ :

$$\Delta \leq floor\left(\frac{3n(n-1)+1}{3}\right) = n(n-1)$$

Taking into account that Δ is linearly decreased with *k* of *n* = 3 · *k* + 2, we get

$$\Delta = n(n-1) - \frac{n-2}{3} - c$$

The correction term c is derived from a discrete order, say n = 5, where we already found that Δ = 18:

$$\Delta\Big|_{n=5} = 18 = 5(5-1) - \frac{5-2}{3} - c = 20 - 1 - c$$

from which c = 1.

This Δ is valid for *n* = 2 mod 3.
Similar formulas can be found for *n* = 0 mod 3 and *n* = 1 mod 3:

$$\Delta\Big|_{n=0 \bmod 3} = n(n-1) - \frac{n}{3}$$

and

$$\Delta\Big|_{n=1 \bmod 3} = n(n-1) - \frac{n-1}{3}$$

together with

$$\Delta\Big|_{n=2 \bmod 3} = n(n-1) - 1 - \frac{n-2}{3}$$

we can write

$$\Delta(n) = n(n-1) - floor\left(\frac{n+1}{3}\right)$$

Table of the first few values of n:

n	s(n)	Δ(n)
2	7	1
3	19	5
4	37	11
5	61	18
6	91	28

See more values under the OEIS sequence A240438!

For a «perfect» OEIS sequence we would like to let it start with $n = 1$. The above formula gives us $\Delta(1) = 0$, which value can be used as a definition for a single cell honeycomb (as e. g. $0! = 1$). But for all those who's mathematical conscience says «nyet!», we let the sequence start with $n = 2$.