ON THE STERN SEQUENCE AND A RELATED SEQUENCE

BY

JENNIFER LANSING

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2014

Urbana, Illinois

Doctoral Committee:

Professor Bruce Berndt, Chair
Professor Bruce Reznick, Director of Research
Professor Alexandru Zaharescu
Professor Kenneth Stolarsky

Abstract

In this dissertation, we discuss properties of the Stern sequence, denoted by $s(n)$, and define a related sequence. First, we give a brief historical background and known results. We then discuss the second and third largest values for the Stern sequence, as well as the asymptotics when a value m will first appear in a row in the diatomic array. We also investigate the distribution of values for the Stern sequence, as well as the gaps of the ordered values from a row.

After this, we investigate the properties of the related sequence called $w(n) := \frac{1}{2}s(3n)$. We give recurrences for the sequence and find generalized recurrences and a reduction formula. We attempt to find a combinatorial interpretation for $w(n)$, as well as a generating function for the sequence. We also find the largest value of $w(n)$ for a row of its triangular array. We consider sums of $w(n)$ and the average order of magnitude, which is the same average order of the Stern sequence. We also examine the greatest common divisor of consecutive terms, as well as the sequence $w(n)$ modulo 2. Finally, we define a polynomial analogue and investigate some of its properties.

I dedicate this dissertation to my family, husband, and God.

Acknowledgments

The author acknowledges support from National Science Foundation grants DMS 08-38434"EMSW21-MCTP: Research Experience for Graduate Students" and DMS 0838434 "EMSW21MCTP: Research Experience for Graduate Students."

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction to the Stern sequence

In this chapter, we give a historical background of the Stern sequence, as well as discuss some more modern results. We then summarize some important properties, and introduce a related sequence.

1.1 A Historical Background

In his 1858 paper [34], Stern investigated the properties of a sequence constructed in a similar way to Pascal's triangle. He constructed a triangular array, called the diatomic array, by taking two values a and b, which form the first row, and then the next row is formed by rewriting the previous row and inserting the sum $a+b$ between its summands. Eisenstein was originally interested in sequences of this type, and asked Stern to consider them as well. Stern studied the properties of this general array, but he also examined the case where $a = b = 1$. The diatomic array with initial values of $(1, 1)$ is given in Table 1.1. In Table 1.2 the

															1		1															
															1	2	1															
														1	3	2	3	1														
												1	4	3	5	2	5	3	4	1												
								1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1								
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11	3	10	7	11	4	9	5	6	1

Table 1.1: Diatomic Array for $s(n)$

starting values are $(0, 1)$, and the array looks more like a sequence since the previous row appears at the start of the next row. In functional notation, the Stern sequence is denoted $s(n)$, and the r-th row of the $(0, 1)$ array in Table 1.2 is given by $s(n)$ for $0 \leq n \leq 2^r$.

															0		1															
															0	1	1															
														0	1	1	2	1														
												0	1	1	2	1	3	2	3	1												
								0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1								
0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5	4	7	3	8	5	7	2	7	5	3	3	7	4	5	1

Table 1.2: $(0, 1)$ Diatomic Array

The r-th row of the $(1,1)$ diatomic array is given by $s(n)$ for $2^r \leq n \leq 2^{r+1}$. We will focus on the array starting with $(1,1)$, and this will be the assumed reference below if a specific reference is not given.

Stern proved numerous properties of the Stern sequence, and these are also summarized in D.H. Lehmer's paper [24] from 1929. We also list some of the properties for handy reference:

- The number of terms in the r-th row is $2^r + 1$, and their sum is $3^r + 1$.
- The number of terms up to and including the r-th row is $2^{r+1} + r$ and their sum is $\frac{1}{2}(3^{r+1} + 1) + r$.
- The average value of the r-th row is approximately $(3/2)^r$.
- The diatomic array is symmetric: the n-th term in the r-th row is equal to the $(2^r + 2 - n)$-th term.
- In the sequence of terms, $(s(n) + s(n+2))/s(n+1)$ is always an integer.
- Two consecutive terms are relatively prime.
- The pair (a, b) occurs at most once in the diatomic array.
- If $\gcd(a, b) = 1$, then the pair (a, b) appears in the line whose whose number is one less than the sum of continuants in the expansion of a/b in a regular continued fraction.
- The number of times an element m appears in the $(m-1)$-th and all succeeding rows is $\phi(m)$.
- The number p is a prime if and only if it appears $(p-1)$ times in the $(p-1)$-th row.

In 1878, Lucas [26] studied Stern's sequence, and noted that the largest value in the r-th row is F_{r+2}, occurring roughly at one third and two thirds of the way in the row. For more on this topic, see Chapter 2.

In 1861, a French clockmaker named Brocot [4] independently created an array of fractions representing gear ratios. This array was very similar to Stern's array, and the fractions correspond to $s(n)/s(2^r - n)$. The connection between these two series was known in the 19th century. In Dickson's *History of the Theory of Numbers* from 1919 [10, p. 156], the sequences that Stern and Brocot studied are listed right next to each other, with Lucas's results on the maximum value following right after. De Rham [9], in his 1947 paper, also discussed Stern's sequence in connection with Brocot's series, as well as Minkowski's ?-function, and these are related to a geometry problem. De Rham also gave recurrence relations for the Stern sequence:

$$s(2n) = s(n), \quad s(2n+1) = s(n+1) + s(n), \quad \text{with } s(0) = 0 \text{ and } s(1) = 1. \tag{1.1}$$

Hermes, a German mathematician, considered sequences of the type that Stern investigated. In his 1894 paper [18], Hermes considered a sequence which turned out to be $s(2n+1)$ and found that this was connected to the binary representation of numbers. This information is essentially mentioned in Dickson's *History of of the Theory of Numbers* ([10, p. 158]). Then in 1902, Bachmann [3], in his book *Niedere Zahlentheorie*, summarized many properties of the Stern sequence and examined them in the context of the Euclidean algorithm and continued fractions.

1.2 Modern Approaches

In 1962-1965, Carlitz independently discovered the Stern sequence, but in the context of Stirling numbers of the second kind. In his first paper [6], Carlitz mentioned a function $\theta_0(n)$, which happens to be $s(n+1)$. The interpretation for this sequence is the number of odd coefficients in a polynomial related to polynomials of Stirling numbers of the second kind. He then gave the generating function

$$\prod_{n=0}^{\infty} \left(1 + x^{2^n} + x^{2^{n+1}}\right) \tag{1.2}$$

for the sequence. His second paper [7] more specifically studied $\theta_0(n)$. He gave the generating function, as well as the combinatorial interpretation that $\theta_0(n)$ gives the number of binary partitions of an integer n, with each part appearing at most twice. He also gives the reduction formula which is equivalent to

$$s(2^r n + 1) = rs(n) + s(n+1),$$

with the following properties as consequences:

$$s(2^r n) = s(n), \quad s(2^r) = 1, \quad \text{and} \quad s(2^r + 1) = r + 1.$$

His third paper [8] still discussed some properties of $\theta_0(n)$, but he focused more on Stirling numbers of the second kind. Then in 1969, Lind [25] presented the connection between Stern's sequence, D.H. Lehmer's paper, and the work of Carlitz. The paper gives a brief summary of known properties for the Stern sequence. Around 1976, Dijkstra [11] also independently discovered Stern's sequence, calling it $fusc(n)$.

There have been numerous papers on further properties of the Stern sequence. In 1990, Reznick [28] discussed binary partition functions, and connected these to the Stern sequence. This gave a combinatorial interpretation of the Stern sequence, as well as the generating function given in (1.2). In 2000, Calkin and Wilf [5] discussed a tree of fractions (which is essentially the Stern-Brocot array), which gives an explicit enumeration of the positive rationals. However, in 1877 Halphen [17] noted that the Stern-Brocot array gave every reduced positive rational exactly once. Since this paper was in French, it is not clear how well known this result was. In the 1994 edition of their book *Concrete Mathematics*, Graham, Knuth, and Patashnik [16, p. 116-117] discuss the Stern-Brocot array and mention that all possible fractions appear exactly once. While Stern had already proven the necessary information used in the proof of the enumeration in [5], Stern's paper was about 15 years before Cantor came up with the question of enumeration. In [27], Reznick also noted that the Stern sequence gave an explicit enumeration of the positive rationals, and gave many other properties of the Stern sequence as well.

A polynomial analogue was also introduced by Stolarsky and Dilcher (see [12, 13]). They defined a polynomial analogue by

$$a(2n, x) = a(n, x^2), \quad a(2n+1, x) = xa(n, x^2) + a(n+1, x^2) \quad \text{for } n \geq 1,$$

with $a(0, x) = 0$ and $a(1, x) = 1$. The first few polynomials are

$$1,\ 1,\ 1+x,\ 1,\ 1+x+x^2,\ 1+x^2,\ 1+x+x^3,\ 1,\ 1+x+x^2+x^4.$$

This is called an analogue of the Stern sequence because $a(n, 1) = s(n)$. Klavžar, Milutinović, and Petr [22] defined a different polynomial analogue:

$$S(2n, x) = xS(n, x), \quad \text{and} \quad S(2n+1, x) = S(n, x) + S(n+1, x) \quad \text{for } n \geq 1,$$

with $S(0, x) = 0$ and $S(1, x) + 1$. The first few polynomials are

$$0,\ 1,\ x,\ 1+x,\ x^2,\ 1+2x,\ x(1+x),\ 1+x+x^2.$$

In Chapter 11, we discuss more about polynomial analogues.

The Stern sequence can also be found in the online encyclopedia of integer sequences as entry A002487 [32].

1.3 More Properties of the Stern Sequence

The Stern sequence satisfies the recurrence relations

$$s(2n) = s(n), \quad s(2n+1) = s(n+1) + s(n), \quad \text{with } s(0) = 0 \text{ and } s(1) = 1.$$

Since $s(2n) = s(n)$, this recurrence implies $s(2^k n) = s(n)$, and we also have that $s(2^k) = 1$. The Stern sequence also satisfies another recurrence formula, given in [27]: for $0 \le j \le 2^r$, we have

$$s(2^r n \pm j) = s(2^r - j)s(n) + s(j)s(n \pm 1). \tag{1.3}$$

A consequence of this formula is

$$s(2^r - 1) = r \quad \text{and} \quad s(2^r + 1) = r + 1. \tag{1.4}$$

Also note that the Stern sequence follows a see-saw pattern in values:

$$s(2n) < s(2n+1) \quad \text{and} \quad s(2n+2) < s(2n+1). \tag{1.5}$$

Since $s(2n+1) = s(n+1) + s(n)$, we have $s(2n+1) > s(n) = s(2n)$ and $s(2n+1) > s(n+1) = s(2n+2)$.

The Stern sequence has lots of symmetry, especially when written in rows with the r-th row consisting of the elements

$$s(2^r), s(2^r + 1), \ldots, s(2^{r+1}).$$

This symmetry is referenced in Stern's paper, as well as Lehmer's paper. However, we define the symmetry in a different way than what was given in the list of properties of [24]. Define $n^* = 3\cdot 2^r - n$ for $2^r < n < 2^{r+1}$. In Table 1.1, we see these rows have reflectional symmetry, which shows that $s(n) = s(n^*)$. There is also a more subtle type of symmetry. Consider the binary representation for n and define $\overleftarrow{n}$ to be the reversal of the binary digits of n. For example, if $n = 19 = [10011]_2$, this is in row $r = 4$, and we have $\overleftarrow{19} = [11001]_2 = 25$, $n^* = 3 \cdot 2^4 - 19 = 29 = [11101]_2$, and $\overleftarrow{n^*} = [10111]_2 = 23$. We have that $s(19) = s(29) = 7 = s(23) = s(25)$. In fact, this holds true for all n, so we have

$$s(n) = s(n^*) = s(\overleftarrow{n}) = s(\overleftarrow{n^*}).$$

Reznick [29] has shown $\overleftarrow{n^*} = \overleftarrow{n}^*$ and that $s(n) = s(\overleftarrow{n})$. These two symmetries form a nice group with four elements, and these elements will always have the same Stern value. However, there are two special cases where there are only two elements: when n is symmetric in binary, and when $\overleftarrow{n} = n^*$.

Stern also noted every third term in the array was even, and in terms of the function, this means $s(3n)$ is always even.

1.4 Introduction to $w(n)$

Looking at $s(3n)$, which is always even, we can ask many questions. In this dissertation, we study the related sequence defined by

$$w(n) := \frac{1}{2}s(3n).$$

How does this sequence behave? It has some similarities to the Stern sequence, such as symmetry of terms in a row in the diatomic array, the same average order of magnitude, and that the sum over powers of 2 is a power of 3. However, $w(n)$ has a much more complicated structure; it has no simple generating function and the recursive definition is not as short. We now show the sequence $w(n)$ can be defined independently of $s(n)$.

Theorem 1.4.1. *Let $w(0) = 0$, $w(1) = 1$ and $w(3) = 2$. For $n \geq 1$, we have*

$$w(2n) = w(n),$$

$$w(8n \pm 1) = w(4n \pm 1) + 2w(n), \tag{1.6}$$

$$w(8n \pm 3) = w(4n \pm 1) + w(2n \pm 1) - w(n). \tag{1.7}$$

Table 1.3 gives a comparison of the first 64 values of $s(n)$ and $w(n)$. Similar to the Stern sequence, we

Table 1.3: Values for $s(n)$ and $w(n)$

n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$
1	1	1	17	5	6	33	6	8	49	9	13
2	1	1	18	4	4	34	5	6	50	7	9
3	2	2	19	7	5	35	9	9	51	12	12
4	1	1	20	3	2	36	4	4	52	5	5
5	3	2	21	8	3	37	11	7	53	13	8
6	2	2	22	5	3	38	7	5	54	8	7
7	3	4	23	7	7	39	10	9	55	11	15
8	1	1	24	2	2	40	3	2	56	3	4
9	4	4	25	7	9	41	11	7	57	10	17
10	3	2	26	5	5	42	8	3	58	7	9
11	5	3	27	8	7	43	13	4	59	11	11
12	2	2	28	3	4	44	5	3	60	4	6
13	5	5	29	7	9	45	12	8	61	9	13
14	3	4	30	4	6	46	7	7	62	5	8
15	4	6	31	5	8	47	9	11	63	6	10
16	1	1	32	1	1	48	2	2	64	1	1

can arrange $w(n)$ into a triangular array, where the k-th row is given by $w(n)$ with $2^k/3 < n < 2^{k+1}/3$, and the convention of when $k = 0$, $n = 0$. The entries in this triangular array are one half the even elements from the diatomic array of the Stern sequence. What properties does this sequence have? How often is $w(n)$

										0										
										1										
										1										
									2	1	2									
								2	4	1	4	2								
					3	2	5	4	6	1	6	4	5	2	3					
3	7	2	9	5	7	4	9	6	8	1	8	6	9	4	7	5	9	2	7	3

Table 1.4: Triangular Array for $w(n)$

even? What is the largest value in a row? What type of recurrence relations does this sequence have? Does it have a nice expression for its generating function?

Looking at Table 1.3, we see many similarities to the Stern sequence. We notice the first five values are the same, and that at powers of 2, the sequence is 1. The Stern sequence has lots of beautiful recurrence relations, and while $w(n)$ is a little more complicated, it has some similar structure. As just mentioned, we have

$$w(2^k n) = \frac{1}{2}s(3 \cdot 2^k n) = \frac{1}{2}s(3n) = w(n).$$

However, the behavior among the odd integers is much harder to discern, and leads us to examine arithmetic progressions of higher powers of 2. We consider these and not other arithmetic progressions, since the behavior of $s(n)$ is better understood when working in terms of powers of 2. For example, we have $w(2n+1) = \frac{1}{2}s(3n+1) + \frac{1}{2}s(3n+2)$, which is not very enlightening. Moving to arithmetic progressions modulo 4, we see

$$w(4n + 1) = \frac{1}{2}s(3n) + s(3n + 1) = w(n) + s(3n + 1), \tag{1.8}$$

$$w(4n + 3) = \frac{1}{2}s(3n + 3) + s(3n + 2) = w(n + 1) + s(3n + 2), \tag{1.9}$$

which is better, but still not completely in terms of $w(n)$. Using a more compact notation, we write (1.8) and (1.9) as

$$w(4n \pm 1) = w(n) + s(3n \pm 1). \tag{1.10}$$

In better hopes of finding a recurrence relation for $w(n)$, we move to arithmetic progressions modulo 8. Using the definition and recurrence relations for $s(n)$, and rearranging the relations in (1.8) and (1.9) to substitute in for $s(3n + 1)$ and $s(3n + 2)$, we obtain the recurrence relations in (1.6) and (1.7), which completes the proof of Theorem 1.4.1. We will continue our discussion of properties of $w(n)$ starting in Chapter 5, but more specifically, we will return to recurrence relations in Chapter 7.

Chapter 2

Maximum Values for the Stern Sequence

The maximum value of $w(n)$ in a row depends on the three largest values of the Stern sequence. We first consider the three largest values taken by $s(n)$, and then in Chapter 6 we consider the maximum value for $w(n)$. However, it is surprising that at least the second largest value has not been previously been considered in the literature.

2.1 Largest Value for $s(n)$

We define the following notation.

Definition 2.1.1. Let $L_m(r)$ denote *the* m*-th largest distinct value in the* r*-th row of the Stern sequence.* The maximum of the r-th row is denoted by $L_1(r)$, the second largest value is denoted by $L_2(r)$, and so forth.

In Table 2.1, we list the maximum of the first 12 rows of the Stern sequence. Lucas [26] observed the

Table 2.1: Largest Values of $s(n)$ in rows

row r	n	$L_1(r)$
0	1	1
1	3	2
2	5,7	3
3	11, 13	5
4	21, 27	8
5	43, 53	13
6	85, 107	21
7	171, 213	34
8	341, 427	55
9	683, 853	89
10	1365, 1707	144
11	2731, 3413	233

maximum of each row of the Stern sequence, when written in the diatomic array, is a Fibonacci number. We restate this theorem, originally proven by Lucas [26] and then later by Lehmer [24] as well.

Theorem 2.1.2. *For all $r \geq 0$, we have $L_1(r) = F_{r+2}$. Moreover, this maximum occurs for the values $n_r = (4 \cdot 2^r - (-1)^r)/3$, as well as $n_r{}^* = (5 \cdot 2^r + (-1)^r)/3$ by symmetry.*

This theorem is proved by an easy induction, which we omit. It is also important to note the relationship between the position of the maximum of the r-th row and that of the $(r-1)$-th row. We have

$$n_r = 2n_{r-1} - (-1)^r, \tag{2.1}$$

which means the maximum for the r-th row will alternate between appearing on the left or right side of the previous maximum in the diatomic array. It is also important to note that $n_r^* = 2n_{r-1}^* + (-1)^r$, by the mirror symmetry of the second half of the row.

Remark 1. There are only two values of n that give the maximum for $s(n)$. For even rows, we have $n = \overleftarrow{n}$ and $n^* = \overleftarrow{n^*}$. For odd rows, we have $\overleftarrow{n} = n^*$ and $n = \overleftarrow{n^*}$.

2.2 Second Largest Value for $s(n)$

Using Mathematica, we can easily compute the second largest value in a particular row of the Stern sequence. The second largest values are given in Table 2.2, and they follow a Fibonacci recurrence relation, $L_2(r) = L_2(r-1) + L_2(r-2)$, for $r \geq 6$. However, starting in the 4th row, there are 4 occurrences of the

Table 2.2: Second Largest Values of $s(n)$ in rows

row r	n	$L_2(r)$
1	2, 4	1
2	6	2
3	9, 15	4
4	19, 23, 25, 29	7
5	45, 51	12
6	83, 91, 101, 109	19
7	173, 181, 203, 211	31
8	339, 363, 405, 429	50
9	685, 725, 811, 851	81
10	1363, 1451, 1621, 1709	131
11	2733, 2901, 3243, 3411	212
12	5459, 5803, 6485, 6829	343

second largest value. Two of them can be seen as adding the second largest values from the two preceding rows, $L_2(r-1) + L_2(r-2)$, while the other two can be seen as a linear combination of previous maximum values, $2L_1(r-2) + L_1(r-4)$.

Example 1. First consider the eighth row. By computing values and using Table 2.2, we see

$$s(363) = s(181) + s(182) = s(181) + s(91) = 31 + 19 = L_2(7) + L_2(6) = 50 = L_2(8).$$

Then using Table 2.1, we find

$$s(339) = 2s(85) + s(84) = 2s(85) + s(21) = 2 \cdot 21 + 8 = 2L_1(6) + L_1(4) = 50 = L_2(8).$$

The second way of obtaining the second largest value, from $2L_1(r-2) + L_1(r-4)$, will occur either 2 to the left or right of where $L_1(r)$ occurs in the row. The Stern sequence achieves these second largest values for explicitly describable n.

Definition 2.2.1. For $r \geq 4$, let

$$n_{2,1}(r) := \frac{17 \cdot 2^{r-2} - (-1)^{r-1}}{3} \quad \text{and} \quad n_{2,2}(r) := \frac{16 \cdot 2^{r-2} - 7(-1)^r}{3}.$$

By the symmetry defined earlier, we have

$$n_{2,1}(r)^* = \frac{19 \cdot 2^{r-2} + (-1)^{r-1}}{3} \quad \text{and} \quad n_{2,2}(r)^* = \frac{20 \cdot 2^{r-2} + 7(-1)^r}{3}.$$

The order of these n is

$$n_{2,2}(r) < n_{2,1}(r) < n_{2,1}(r)^* < n_{2,2}(r)^*, \quad \text{for } r \geq 6.$$

Note that $n_{2,1}(r)$ and $n_{2,2}(r)$ coalesce at $r = 5$ so that there are only two occurrences of $L_2(5)$.

We observe that $s(n_{2,1}(r))$ and $s(n_{2,1}(r)^*)$ give the second largest value in the r-th row as a sum of preceding second largest values, and $s(n_{2,2}(r))$ and $s(n_{2,2}(r)^*)$ give the second largest value of the r-th row as a linear combination of previous maximum values. Also note that $n_{2,1}(r)$ has a similar recurrence relation as n_r in (2.1):

$$n_{2,1}(r) = \frac{17 \cdot 2^{r-2} - (-1)^{r-1}}{3} = \frac{17 \cdot 2^{r-2} + 2(-1)^{r-1} - 3(-1)^{r-1}}{3} = 2n_{2,1}(r-1) + (-1)^r. \tag{2.2}$$

It is also useful to note

$$\begin{aligned} n_{2,2}(r) &= \frac{16 \cdot 2^{r-2} - 7(-1)^r}{3} = 4 \cdot \frac{2^r - (-1)^r}{3} - (-1)^r = 4n_{r-2} - (-1)^r. \\ &= n_r - 2(-1)^r. \end{aligned} \tag{2.3}$$

The last equality makes it explicit that the second way of obtaining the second largest value, from $2L_1(r-2)+L_1(r-4)$, will occur either two to the left or right of where $L_1(r)$ occurs in the row.

Theorem 2.2.2. *We have the following.*

(i) $L_2(r) = F_{r+2} - F_{r-3} = L_1(r) - F_{r-3}$, *for* $r \geq 4$.

(ii) $s(n_{2,1}(r)) = s(n_{2,1}(r)^*) = s(n_{2,2}(r)) = s(n_{2,2}(r)^*) = L_2(r)$, *for* $r \geq 4$.

(iii) *For* $n_{2,1}(r)$ *and* $n_{2,1}(r)^*$, $L_2(r)$ *arises from the sum* $L_2(r-1) + L_2(r-2)$, *for* $r \geq 6$.

(iv) *For* $n_{2,2}(r)$ *and* $n_{2,2}(r)^*$, $L_2(r)$ *arises from the sum* $2L_1(r-2) + L_1(r-4)$, *for* $r \geq 8$.

Proof. The induction base is clear and easily verified. Assume (i)-(iv) hold for all rows before and including the r-th row, with $r \geq 8$.

We first show that besides the maximum value, there is nothing larger than $L_2(r) + L_2(r-1)$ in the $(r+1)$-th row. If $2k \in [2^{r+1}, 2^{r+2}]$, then using the induction hypothesis from part (i), we find that for all the even values in the $(r+1)$-th row

$$s(2k) = s(k) \leq L_1(r) = F_{r+2} < F_{r+2} + 2F_{r-1} = F_{r+3} - F_{r-2} = L_2(r) + L_2(r-1).$$

We now want to obtain a bound on the odd values in the $(r+1)$-th row, and more specifically $s(4k \pm 1)$. Now let $k \in [2^{r-1}, 2^r]$ such that $4k \pm 1 \in (2^{r+1}, 2^{r+2})$. We examine some special cases first. If $k = n_{r-1}$, or $k = n^*_{r-1}$, we see

$$s(2n_{r-1} - (-1)^r) = s(n_r) = L_1(r) = s(n^*_r) = s(2n^*_{r-1} + (-1)^r),$$

which implies $s(2n_{r-1} + (-1)^r) < L_1(r)$ and $s(2n^*_{r-1} - (-1)^r) < L_1(r)$. Similarly, if $k = 2n_{r-2}$ or $k = 2n^*_{r-2}$, we have

$$s(2 \cdot 2n_{r-2} + (-1)^r) = s(n_r) = L_1(r) = s(n^*_r) = s(2 \cdot 2n^*_{r-2} - (-1)^r),$$

which means $s(2 \cdot 2n_{r-2} - (-1)^r) < L_1(r)$ and $s(2 \cdot 2n^*_{r-2} + (-1)^r) < L_1(r)$. So except for the special cases where $k = n_{r-1}$, n^*_{r-1}, $2n_{r-2}$, or $2n^*_{r-2}$, we have $s(2k \pm 1) < L_1(r)$, and therefore $s(2k \pm 1) \leq L_2(r)$. Then we have

$$s(4k \pm 1) = s(2k) + s(2k \pm 1) = s(2k \pm 1) + s(k) \leq L_2(r) + L_2(r-1),$$

We now examine the special cases more closely. If $k = n_{r-1}$, we have

$$s(4n_{r-1} - (-1)^r) = s(2n_{r-1}) + s(2n_{r-1} - (-1)^r) = s(n_{r-1}) + s(n_r) = L_1(r+1),$$

but we can disregard this (largest) value, since we are looking for the second largest value. Now, we also have

$$\begin{aligned}
s(4n_{r-1}+(-1)^r) &= s(2n_{r-1})+s(2n_{r-1}+(-1)^r)\\
&= s(n_{r-1})+s(n_{r-1})+s(n_{r-1}+(-1)^r)\\
&= 2s(n_{r-1})+s(2n_{r-2}+2(-1)^r)\\
&= 2s(n_{r-1})+s(n_{r-2}+(-1)^r)\\
&= 2s(n_{r-1})+s(2n_{r-3}+-(-1)^r+(-1)^r)\\
&= 2s(n_{r-1})+s(n_{r-3})\\
&= 2L_1(r-1)+L_1(r-3). \qquad (2.4)
\end{aligned}$$

Note that by symmetry we have $s(4n^*_{r-1}-(-1)^r) = 2L_1(r-1)+L_1(r-3)$ and $s(4n^*_{r-1}+(-1)^r) = L_1(r)$. Lastly, if $k = 2n_{r-2}$ we have

$$\begin{aligned}
s(4\cdot 2n_{r-2}\pm 1) &= 2s(n_{r-2})+s(2n_{r-2}\pm 1)\\
&= \begin{cases} 2s(n_{r-2})+s(2n_{r-2}+(-1)^r) \\ 2s(n_{r-2})+s(2n_{r-2}-(-1)^r) \end{cases}\\
&= \begin{cases} 2s(n_{r-2}+s(n_{r-1}) = 2L_1(r-2)+L_1(r-1) \\ 3s(n-r-2)+s(n_{r-4}) = 3L_1(r-2)+L_1(r-4) \end{cases}\\
&< 2L_1(r-1)+L_1(r-3).
\end{aligned}$$

So then we only need compare $2L_1(r-1)+L_1(r-3)$ to $L_2(r)+L_2(r-1)$. However, using part (i) from the induction hypothesis, we have

$$\begin{aligned}
L_2(r)+L_2(r-1) &= F_{r+2}-F_{r-3}+F_{r+1}-F_{r-4}\\
&= F_{r+3}-F_{r-2} = L_1(r+1)-L_1(r-4)\\
&= 2L_1(r-1)+L_1(r-3).
\end{aligned}$$

Thus, all elements in the $(r+1)$-th row, besides the maximum, are less than or equal to $L_2(r)+L_2(r-1)$, and so we have

$$L_2(r+1) = F_{r+3} - F_{r-2} = L_2(r) + L_2(r-1) = 2L_1(r-1) + L_1(r-3).$$

All that is left then is to verify that the second largest values occur for the n given earlier. Evaluating $s(n_{2,1}(r+1))$ and using (2.2), we have

$$\begin{aligned} s(n_{2,1}(r+1)) = s(2n_{2,1}(r) + (-1)^{r+1}) = s(n_{2,1}(r)) + s(n_{2,1}(r) - (-1)^r) = s(n_{2,1}(r)) + s(2n_{2,1}(r-1)) \\ = s(n_{2,1}(r)) + s(n_{2,1}(r-1)) \\ = L_2(r) + L_2(r-1). \end{aligned}$$

We also note that by (2.3) we have $n_{2,2}(r+1) = 4n_{r-1} + (-1)^r$, and in (2.4) we see that $s(4n_{r-1} + (-1)^r)$ also gives $L_2(r)$. Therefore, the second largest values occur where we expect. Finally, by the symmetry of the Stern sequence in rows, we have

$$s(n_{2,1}(r+1)^*) = s(n_{2,1}(r+1)) = L_2(r+1) = s(n_{2,2}(r+1)) = s(n_{2,2}(r+1)^*). \qquad \square$$

Table 2.3: Binary representation for $n_{2,1}(r)$, $n_{2,1}(r)^*$, $n_{2,2}(r)$, and $n_{2,2}(r)^*$

row r	$n_{2,2}(r)$	$n_{2,1}(r)$	$n_{2,1}(r)^*$	$n_{2,2}(r)^*$
4	10011	10111	11001	11101
5	101101	101101	110011	110011
6	1010011	1011011	1100101	1101101
7	10101101	10110101	11001011	11010011
8	101010011	101101011	110010101	110101101
9	1010101101	1011010101	1100101011	1101010011
10	10101010011	10110101011	11001010101	11010101101
11	101010101101	101101010101	110010101011	110101010011
12	1010101010011	1011010101011	1100101010101	1101010101101
13	10101010101101	10110101010101	11001010101011	11010101010011

Remark 2. Table 2.3 gives the binary representation of the n which give $L_2(r)$, and we see several striking patterns. We see that $\overleftarrow{n_{2,1}(r)} = n_{2,2}(r)^*$ and $\overleftarrow{n_{2,2}(r)} = n_{2,1}^*$ for even $r \geq 6$. For odd $r \geq 7$, we have $\overleftarrow{n_{2,1}(r)} = n_{2,2}(r)$ and $\overleftarrow{n_{2,1}^*} = n_{2,2}^*$. Also note that for $r = 5$, the values $n_{2,1}(5)$ and $n_{2,2}(5)$ coalesce, and all of the values are symmetric, so that $n_{2,1}(5) = n_{2,2}(5) = \overleftarrow{n_{2,1}(5)} = \overleftarrow{n_{2,2}(5)}$.

2.3 Third Largest Value for $s(n)$

The third largest values in a row for $s(n)$, given in Table 2.4, also satisfy a Fibonacci recurrence. This recurrence starts in the 10th row, and rows 8 and 9 give the two initial values. Similar to the second largest

Table 2.4: Third Largest Values of $s(n)$ in rows

row r	n	$L_3(r)$
1	N/A	N/A
2	4	1
3	10, 14	3
4	17, 22, 26, 31	5
5	37, 41, 55, 59	11
6	75, 87, 105, 117	18
7	165, 219	30
8	331, 347, 421, 437	49
9	693, 843	80
10	1355, 1387, 1685, 1717	129
11	2741, 2773, 3371, 3403	209
12	5451, 5547, 6741, 6837	338

value in a row, there are 4 occurrences of the third largest value. By symmetry, two of them come from $L_3(r-1)+L_3(r-2)$, and the other two come from the sum of $(2L_1(r-4)+L_1(r-6))+(3L_1(r-4)+2L_1(r-6))$, which adds to $5L_1(r-4)+3L_1(r-6)$.

The values of n where the third largest values occur also follow a pattern, starting in the eighth row. For the outside left and right values, it appears that the next value is two times the previous value plus or minus 31. The inside right and left values follow the pattern, two times the previous value plus or minus one. Taking this recurrences and reiterating them, we obtain the following apparent closed form for these values.

Definition 2.3.1. For $r \geq 8$, let

$$n_{3,1}(r) := \frac{64 \cdot 2^{r-4} - 31(-1)^r}{3} \quad \text{and} \quad n_{3,2}(r) := \frac{65 \cdot 2^{r-4} + (-1)^r}{3}.$$

We note

$$n_{3,1}(r)^* = \frac{80 \cdot 2^{r-4} + 31(-1)^r}{3} \quad \text{and} \quad n_{3,2}(r)^* = \frac{79 \cdot 2^{r-4} - (-1)^r}{3}.$$

We will show that $n_{3,1}(r)$, $n_{3,1}(r)^*$, $n_{3,2}(r)$, and $n_{3,2}(r)^*$ give the third largest values for $s(n)$ in the r-th row. It is useful to note

$$n_{3,1}(r) = \frac{4 \cdot 2^r - (-1)^r - 30(-1)^r}{3} = n_r - 10(-1)^r. \tag{2.5}$$

This tells us that the first (and by symmetry, the last) occurrence of the third largest value in the row will be either 10 to the left or 10 to the right, depending on the parity of the row, of where the largest values occur. It is also helpful to remark that $n_{3,2}(r)$ satisfies the recurrence relations

$$n_{3,2}(r) = 2n_{3,2}(r-1) + (-1)^r \quad \text{and} \quad n_{3,2}(r) - (-1)^r = 2n_{3,2}(r-1). \tag{2.6}$$

We also have

$$n_{3,1}(r) < n_{3,2} < n_{3,2}(r)^* < n_{3,1}(r)^*, \quad \text{ for } r \geq 8.$$

Theorem 2.3.2. *We have the following:*

(i) $L_3(r) = L_2(r) - F_{r-7} = F_{r+1} + 5F_{r-4} = L_1(r) - 3F_{r-5}$*, for* $r \geq 8$.

(ii) $s(n_{3,1}(r)) = s(n_{3,1}(r)^*) = s(n_{3,2}(r)) = s(n_{3,2}(r)^*) = L_3(r)$*, for* $r \geq 8$.

(iii) *For* $n_{3,1}(r)$ *and* $n_{3,1}(r)^*$*,* $L_3(r)$ *arises from the sum* $5L_1(r-4) + 3L_1(r-6)$*, for* $r \geq 8$.

(iv) *For* $n_{3,2}(r)$ *and* $n_{3,2}(r)^*$*,* $L_3(r)$ *arises from the sum* $L_3(r-1) + L_3(r-2)$*, for* $r \geq 10$.

Proof. We proceed by induction. The base case is easily verified. Assume the induction hypotheses hold for all values below $r > 10$. We will first verify that $L_3(r) + L_3(r-1)$ and $5L_1(r-3) + 3L_1(r-5)$ are achieved at the expected values. We will then show that all values in the $(r+1)$-th row, except for $L_1(r+1)$ and $L_2(r+1)$, are less than or equal to $L_3(r) + L_3(r-1) = F_{r+2} + 5F_{r-3}$.

We first verify that $s(n_{3,2}(r+1)) = L_3(r) + L_3(r-1)$. Using (2.6), we have

$$\begin{aligned} s(n_{3,2}(r+1)) &= s(2n_{3,2}(r) - (-1)^r) = s(n_{3,2}(r)) + s(n_{3,2}(r) - (-1)^r) \\ &= L_3(r) + s(2n_{3,2}(r-1)) \\ &= L_3(r) + L_3(r-1). \end{aligned}$$

Now considering $s(n_{3,1}(r+1))$, and using (2.5) and (2.1) we have

$$
\begin{aligned}
s(n_{3,1}(r+1)) &= s(n_{r+1}+10(-1)^r) = s(2n_r+(-1)^r+10(-1)^r) \\
&= s(n_r+5(-1)^r)+s(n_r+5(-1)^r+(-1)^r) \\
&= s(2n_{r-1}-(-1)^r+5(-1)^r)+s(2n_{r-1}+6(-1)^r-(-1)^r) \\
&= s(n_{r-1}+2(-1)^r)+s(n_{r-1}+3(-1)^r)+s(n_{r-1}+3(-1)^r-(-1)^r) \\
&= 2s(n_{r-1}+2(-1)^r)+s(n_{r-1}+3(-1)^r) \\
&= 2s(2n_{r-2}+(-1)^r+2(-1)^r)+s(2n_{r-2}+(-1)^r+3(-1)^r) \\
&= 2s(n_{r-2}+(-1)^r)+2s(n_{r-2}+2(-1)^r)+s(n_{r-2}+2(-1)^r) \\
&= 2s(n_{r-3})+3s(2n_{r-3}+(-1)^r) \\
&= 2s(n_{r-3})+3s(n_{r-3})+3s(n_{r-3}+(-1)^r) \\
&= 5s(n_{r-3})+3s(2n_{r-4}+2(-1)^r) \\
&= 5s(n_{r-3})+3s(n_{r-4}+(-1)^r) \\
&= 5s(n_{r-3})+3s(2n_{r-5}) \\
&= 5L_1(r-3)+3L_1(r-5).
\end{aligned}
$$

We also note by hypotheses (i), we have

$$L_3(r)+L_3(r-1) = F_{r+3}-3F_{r-4} = 5F_{r-1}+3F_{r-3} = 5L_1(r-3)+3L_1(r-5). \tag{2.7}$$

We now show that all values in the $(r+1)$-th row, except for $L_1(r+1)$ and $L_2(r+1)$, are less than or equal to $L_3(r)+L_3(r-1) = F_{r+2}+5F_{r-3}$. First note that for $2k \in [2^{r+1}, 2^{r+2}]$, we have

$$s(2k) = s(k) \le L_1(r) = F_{r+2} < F_{r+2}+5F_{r-3} = L_3(r)+L_3(r-1).$$

For the odd values in the $(r+1)$-th row, we eliminate the cases that anything larger than $L_3(r)+L_3(r-1)$ can come from the first, second or third largest values and a neighbor. Now consider $s(2k+1) = s(k)+s(k+1)$, and let b represent the larger of $s(k)$ and $s(k+1)$, which comes from the r-th row, and c denote the smaller value coming from the $(r-1)$-th row.

If b is $L_1(r)$, then c is either $L_1(r-1)$ or $L_1(r-2)$. However, $L_1(r)+L_1(r-1)$ gives the largest value in the $(r+1)$-th row, so we can ignore this value. We have

$$L_1(r) + L_1(r-2) = F_{r+2} + F_r < F_{r+2} + 5F_{r-3}$$

is too small. So then it must be the case that $b \leq L_2(r)$. Now, there are 2 distinct ways of obtaining $L_2(r)$. If $b = L_2(r)$, then c could be $L_2(r-1)$ or $L_2(r-2)$. But $L_2(r) + L_2(r-1)$ gives $L_2(r+1)$ and we can ignore this value. We also have

$$L_2(r) + L_2(r-2) = F_{r+2} - F_{r-3} + F_r - F_{r-4} = F_{r+2} + F_{r-2} + 2F_{r-4} < F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1),$$

so this value is too small. But if $b = 2L_1(r-2) + L_1(r-4)$, then c is $L_1(r-2)$ or $L_1(r-2) + L_1(r-4)$, and we only need to consider the latter. We see

$$2L_1(r-2) + L_1(r-4) + L_1(r-2) + L_1(r-4) = 3F_r + 2F_{r-2} < F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1),$$

which implies $b \leq L_3(r)$. If $b = L_3(r)$, there are 2 distinct ways of obtaining $L_3(r)$. If $b = L_3(r)$, then c could be $L_3(r-1)$ or $L_3(r-2)$, but we can disregard these because we want to find something larger. If $b = 5L_1(r-4) + 3L_1(r-6)$, then c could be $2L_1(r-4) + L_1(r-6)$ or $3L_1(r-4) + 2L_1(r-6)$. We need only consider the latter, and we see

$$5L_1(r-4) + 3L_1(r-6) + 3L_1(r-4) + 2L_1(r-6) = 8F_{r-2} + 5F_{r-4} < F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1),$$

which implies $b < L_3(r)$. However c might be a large value in the $(r-1)$-the row and make up for b being so small. Now, we know $c < b < L_3(r)$, that c comes from the $(r-1)$-th row, and that $c > L_3(r-1)$. If $c = L_1(r-1)$, then b could only be $L_1(r-2) + L_1(r-3)$ (since we already eliminated $b = L_1(r)$). But then $b + c = 2L_1(r-1) + L_1(r-3) = L_2(r+1)$, and we can also ignore this value. So then c must be $L_2(r-1)$. Then the only possibility for b is $L_2(r-1) + L_2(r-3)$ (as we already eliminated $b = L_2(r)$). By Theorem 2.2.2 (i) and the induction hypothesis (i), we have

$$\begin{aligned} L_2(r+1) - F_{r-6} &= L_2(r) + L_2(r-1) - F_{r-6} \\ &= L_3(r) + F_{r-7} + L_3(r-1) + F_{r-8} - F_{r-6} \\ &= L_3(r) + L_3(r-1). \end{aligned} \tag{2.8}$$

So then using (2.8) we see

$$\begin{aligned} b+c &= 2L_2(r-1)+L_2(r-3) \\ &= 2L_1(r-1)-2F_{r-4}+L_1(r-3)-F_{r-6} \\ &= L_2(r+1)-2F_{r-4}-F_{r-6} \\ &< L_2(r+2)-F_{r-6} = L_3(r)+L_3(r-1). \end{aligned}$$

Finally, we have that c could be $L_2(r-1)=2L_1(r-3)+L_1(r-5)$ and b could be $3L_1(r-3)+2L_1(r-5)$ or $3L_1(r-3)+L_1(r-5)$. Then by (2.7) we see

$$L_3(r)+L_3(r-1)=5L_1(r-3)+3L_1(r-5)>5L_1(r-3)+2L_1(r-5),$$

so that $5L_1(r-3)+2L_1(r-5)$ is too small, while $5L_1(r-3)+3L_1(r-5)$ gives $L_3(r)+L_3(r-1)$. Then this eliminates all possibilities and shows nothing can be between $L_3(r)+L_3(r-1)$ and $L_2(r+1)$, so that $s(2k+1)\le L_3(r)+L_3(r-1)$. Thus we have $L_3(r+1)=L_3(r)+L_3(r-1)$. □

We now search for a relationship between n, n^*, and $\overleftarrow{n}$ for $n_{3,1}$ and $n_{3,2}$. Looking at the binary representations as given in Table 2.5, we see that $n_{3,1}(9)$ is symmetric in its binary representation. For this row, there are only 2 distinct elements, namely, $n_{3,1}$ and $n^*_{3,1}$. Otherwise, the relationships between $n_{3,1}$, $n_{3,2}$, $n^*_{3,1}$, and $n^*_{3,2}$ follow the same patterns as those for the n that give the second largest value. For even rows, we have $\overleftarrow{n_{3,1}}=n^*_{3,2}$ and $\overleftarrow{n_{3,2}}=n^*_{3,1}$. For odd rows, we have $\overleftarrow{n_{3,1}}=n_{3,2}$ and $\overleftarrow{n^*_{3,1}}=n^*_{3,2}$.

Table 2.5: Binary representation for $n_{3,1}(r)$, $n_{3,1}(r)^*$, $n_{3,2}(r)$, and $n_{3,2}(r)^*$

row r	$n_{3,1}(r)$	$n_{3,2}(r)$	$n_{3,2}(r)^*$	$n_{3,1}(r)^*$
8	101001011	101011011	110100101	110110101
9	1010110101	1010110101	1101001011	1101001011
10	10101001011	10101101011	11010010101	11010110101
11	101010110101	101011010101	110100101011	110101001011
12	1010101001011	1010110101011	1101001010101	1101010110101
13	10101010110101	10101101010101	11010010101011	11010101001011
14	101010101001011	101011010101011	110100101010101	110101010110101
15	1010101010110101	1010110101010101	1101001010101011	1101010101001011
16	10101010101001011	10101101010101011	11010010101010101	11010101010110101

Remark 3. Again, note that there are only two occurrences of L_1 in a row, whereas for L_2 and L_3, we have the group of four elements as mentioned in Chapter 1.

Since the next row in the diatomic array is formed by inserting the sum of two consecutive terms in between them, it makes sense that the k-th largest value in the r-th row will satisfy a Fibonacci recurrence. This appears to hold true for the 4th, 5th, and 6th (distinct) largest value in a row, for a sufficiently large row value. We see in Table 2.6 a Fibonacci recurrence starts at the 14th row for L_4, the 18th for L_5, and

Table 2.6: L_m of $s(n)$ in rows

row r	$L_1(r)$	$L_2(r)$	$L_3(r)$	$L_4(r)$	$L_5(r)$	$L_6(r)$
3	5	4	3	2	1	N/A
4	8	7	5	4	3	2
5	13	12	11	10	9	8
6	21	19	18	17	16	15
7	34	31	30	29	27	26
8	55	50	49	47	46	45
9	89	81	80	79	76	75
10	144	131	129	128	123	121
11	233	212	209	208	207	199
12	377	343	338	337	335	322
13	610	555	547	546	545	542
14	987	898	885	883	882	877
15	1597	1453	1432	1429	1428	1427
16	2584	2351	2317	2312	2311	2309
17	4181	3804	3749	3741	3740	3739
18	6765	6155	6066	6053	6051	6050
19	10946	9959	9815	9794	9791	9790
20	17711	16114	15881	15847	15842	15841
21	28657	26073	25696	25641	25633	25632
22	46368	42187	41577	41488	41475	41473
23	75025	68260	67273	67129	67108	67105

the 22nd row for L_6. This leads us to the following conjecture.

Conjecture 2.3.3. *For all $r \geq 4m-2$, the m-th largest distinct value satisfies the recurrence $L_m(r) = L_m(r-1) + L_m(r-2)$.*

Remark 4. By Theorems 2.2.2 (i) and 2.3.2 (i), we have $L_2(r) = F_{r+2} - F_{r-3}$ and $L_3(r) = L_2(r) - F_{r-7}$. Note that $L_3(r) = F_{r+2} - F_{r-3} - F_{r-7}$. Inspecting Table 2.6, we notice the following:

$$L_4(r) = L_3(r) - F_{r-11} = F_{r+2} - F_{r-3} - F_{r-7} - F_{r-11} \quad \text{for } r \geq 12,$$

$$L_5(r) = L_4(r) - F_{r-15} = F_{r+2} - F_{r-3} - F_{r-7} - F_{r-11} - F_{r-15} \quad \text{for } r \geq 16, \text{ and}$$

$$L_6(r) = L_5(r) - F_{r-19} = F_{r+2} - F_{r-3} - F_{r-7} - F_{r-11} - F_{r-15} - F_{r-19} \quad \text{for } r \geq 20.$$

This leads us to another conjecture.

Conjecture 2.3.4. *For $r \geq 4(m-1)$, we have*

$$L_m(r) = L_{m-1}(r) - F_{r-(4m-5)} = F_{r+2} - \sum_{j=2}^{m} F_{r-(4j-5)}.$$

Finally, we examine the limits of the ratios of the first three largest values for $s(n)$ and L_1. We have

$$\begin{aligned}
\lim_{r\to\infty} \frac{L_1(r)}{L_1(r)} &= 1, \\
\lim_{r\to\infty} \frac{L_2(r)}{L_1(r)} &= \lim_{r\to\infty} \left(1 - \frac{F_{r-3}}{F_{r+2}}\right) = 1 - \frac{1}{\phi^5} \approx 0.90983, \\
\lim_{r\to\infty} \frac{L_3(r)}{L_1(r)} &= \lim_{r\to\infty} \left(1 - \frac{3F_{r-5}}{F_{r+2}}\right) = 1 - \frac{3}{\phi^7} \approx 0.896674.
\end{aligned}$$

If Conjecture 2.3.4 is correct, then we have

$$\lim_{k\to\infty} \left(\lim_{r\to\infty} \frac{L_k(r)}{L_1(r)}\right) = 1 - \sum_{j=1}^{\infty} \frac{1}{\phi^{4j+1}} = 1 - \left(1 - \frac{2\sqrt{5}}{5}\right) = \frac{2\sqrt{5}}{5} \approx 0.8944.$$

This means the k-th largest values stay around $0.8944F_{r+2}$. More importantly, this also tells us the number of n with $2^r < n < 2^{r+1}$, such that

$$s(n) > \frac{2\sqrt{5}}{5} F_{r+2}$$

goes to infinity as r goes to infinity. While the number of n grows without bound, it is still small compared to the number of total elements in a row.

Chapter 3

First Appearance of Values in a Row

In his 1858 paper, Stern [34] considered several questions regarding when a number first appears in a row or how often it occurs. These results are also summarized in Lehmer's paper [24]. In the context of the diatomic array, a "new" entry is an entry which comes from the sum of two entries from the previous row. These new entries correspond to $s(n)$ for odd n. Stern also considered how many times a number m will appear in a row, and in which line m will last appear as a new entry.

First, we have m will not appear past the m-th row as a new entry. The last time m will appear as a new entry will be in the $(m-1)$-th row. Since m comes from the sum of two consecutive and relatively prime entries from the previous row, this means there are $\phi(m)$ different ways to obtain m as a new entry. So each m will appear at most $\phi(m)$ times in a row, and for every row after the $(m-1)$-th row, m will appear exactly $\phi(m)$ times. For primes, this means p will occur $p-1$ times in the $(p-1)$-th row, and we can say something a little stronger too.

Theorem 3.0.5. *The number p is a prime if and only if it appears $p-1$ times in the $(p-1)$th row.*

This theorem, proved by Stern, is also mentioned in Dickson's *History* [10] as a test for primality.

In thinking about how soon a number will appear, the Fibonacci numbers appear earliest compared to other numbers of the same size, since they are generally the largest value in a row. This gives us a lower bound for the row in which the number m will appear.

Theorem 3.0.6. *The number m will appear no earlier than in the row given by $\left[\frac{\ln(\sqrt{5}m)}{\ln(\phi)} - 2\right]$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

Proof. In the r-th row, the largest number is F_{r+2}. Using Binet's formula, we have

$$F_r = \frac{\phi^r - \overline{\phi}^r}{\sqrt{5}}.$$

in solving for r, we have

$$\ln F_r = r \ln \phi + \ln\left(1 - \left(\frac{\overline{\phi}}{\phi}\right)^r\right) - \ln\sqrt{5},$$

so that

$$r = \frac{\ln(F_r\sqrt{5})}{\ln\phi} - \frac{1}{\ln\phi}\ln\left(1 - \left(\frac{\overline{\phi}}{\phi}\right)^r\right)$$
$$= \frac{\ln(F_r\sqrt{5})}{\ln\phi} - \frac{1}{\ln\phi} + o(1). \qquad (3.1)$$

If we define $v(m)$ to be a function which gives the row where m first appears, then we have $v(F_r) = r - 2$. Together with (3.1), this means $v(F_r) = r - 2 \approx \frac{\ln(F_r\sqrt{5})}{\ln\phi} - 2$. Then we can expect $v(m)$ to be bounded below by $\left[\frac{\ln(\sqrt{5}m)}{\ln(\phi)}\right] - 2$. □

Figure 3.1 compares $v(m)$ with the lower bound $\frac{\ln(\sqrt{5}m)}{\ln(\phi)} - 2$ for the first 100 values and then also for the first 2^{12} values. Let

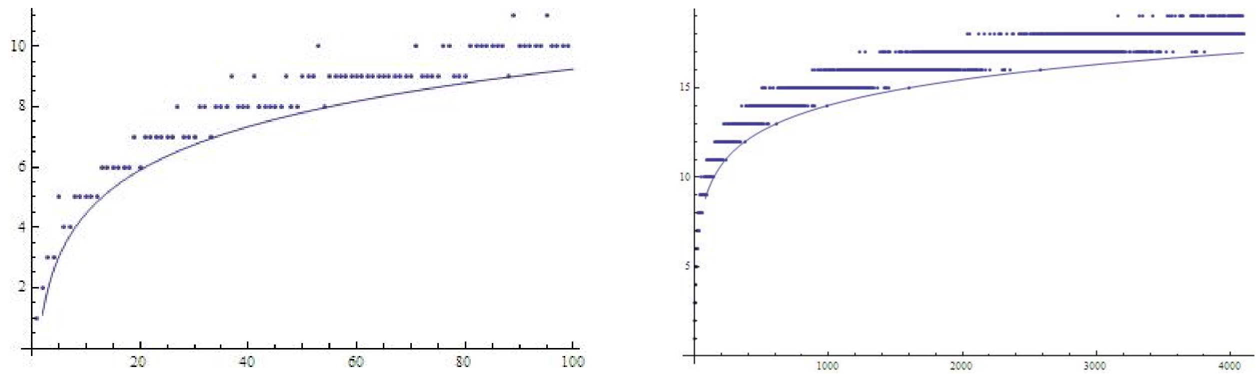


Figure 3.1: $v(m)$ compared to $\frac{\ln(\sqrt{5}m)}{\ln(\phi)} - 2$

$$f(x) := \left[\frac{\ln(\sqrt{5}x)}{\ln(\phi)}\right] - 2.$$

Figure 3.2 graphs the difference $v(m) - f(m)$ for m up to 2^{12}. This shows us that $f(x)$ works as a lower bound, and this had been verified up to $2^{20} - 1$. It appears that a possible upper bound would be the lower bound shifted up by 3, or $f(x) + 3$. Some of the values where the difference is 3 are 1230, 3160, 4470, 5052, 5082, 5190, 5208, 5262, 5280, 5304, 7764, 8022, 8070, 8088, and 8176. Computing up to $m = 2^{20} - 1$, we found the only values that occur in the difference are 0, 1, 2, and 3. There are many values of m which have a distance of 3 from the lower bound, but this seems to be the maximum. This leads us to the following conjecture.

Conjecture 3.0.7. *For all m, we have $f(m) \leq v(m) \leq f(m) + 3$.*

Stern also discovered a way of finding the rows in which a value m could be found. He noted that if a and b are consecutive terms, then the pair (a, b) can occur only once in the diatomic array. By symmetry,

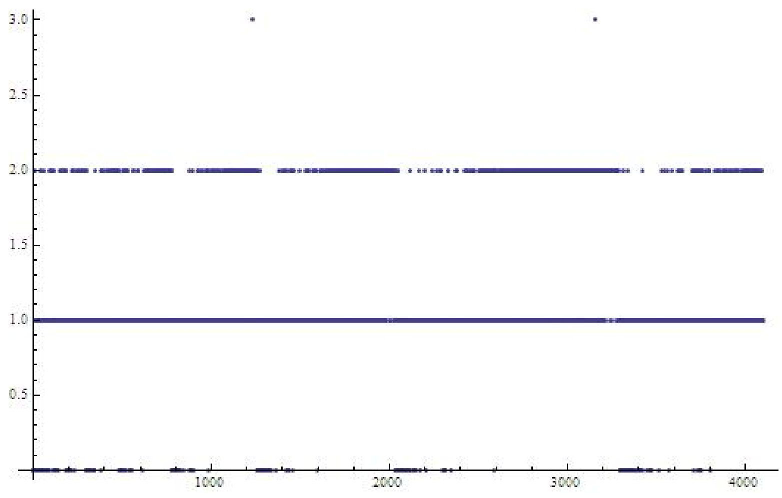


Figure 3.2: $v(m) - f(m)$ for m up to 2^{12}

the sequence (b, a) will appear in the second half of the row. In addition, (a, b) will appear in the line whose number is one less than the sum of the continuants in the regular continued fraction expansion of a/b. In other words, if the continued fraction of $a/b = [a_1, a_2, a_3, \ldots, a_n]$, then (a, b) will appear in the $(a_1 + a_2 + a_3 + \cdots + a_n - 1)$-th row.

We can also rephrase this problem another way. If m is a new entry, then there exist relatively prime a and b such that $a + b = m$. This means m will then appear in the row given by the sum of the continuants from the continued fraction of a/b. In finding all such possible a and b and computing the sum of the continuants of a/b, we can find all the rows in which m will appear. With this approach, Conjecture 3.0.7 can be interpreted as follows:

For fixed m, consider all pairs (a, b) with $\gcd(a, b) = 1$ and $a + b = m$, with the continued fraction expansion $a/b = [a_1, a_2, a_3, \ldots, a_n]$. Then the smallest sum of $a_1 + a_2 + a_3 + \cdots + a_n$ is of the magnitude of $f(m)$. In other words,

$$\min_{(a,b)}(a_1 + a_2 + a_3 + \cdots + a_n) = f(m) + O(1).$$

We know roughly in which row a value m will appear and the last row in which it will appear as a new entry. Future questions to be investigated include what happens in between. In which row will the value m appear for the second time? Is there a distribution for the rows in which the value m will appear as a new entry?

Chapter 4

The Distribution of Values

This chapter is based on the author's publication [23], "Distribution of values of the the binomial coefficients and the Stern sequence."

4.1 Background & Motivation

Historically there have been two stages to understanding the distribution of sequences, from a number theoretical standpoint. The first and more classical stage involves examining the distribution of values in a sequence. Hermann Weyl [37] made many advances in this area by proving certain sequences are uniformly distributed. For example, Weyl proved that if α is an irrational number, then for any positive integer d, the sequence $\{\alpha n^d\}$ is uniformly distributed.

Once the distribution of values for a sequence is well understood, then a more modern approach is to consider the distribution of spacings between consecutive terms. Some famous results in this area include work by Hooley [19, 20] and Gallagher [14] on the distribution of gaps between consecutive primes. The limiting distribution for average gaps between primes is Poisson, and the spacings of fractional parts of lacunary sequences (see [30]) are also Poisson. Another famous result in this area is the Steinhaus Conjecture, also known as the Three Gap Theorem. For example, for any irrational α, the gaps between consecutive terms after ordering the sequence $\{\alpha n\}$ up to a certain N, will only take 3 values, one of which is the sum of the other two (see [33, 36]). For a more complete background on the distribution of spacings, see the first few pages of [2].

The distribution of values is well understood for many sequences, but what about the Stern sequence? With the goal in mind of studying the distribution of spacing of the Stern sequence, we first need to understand the distribution of values. Understanding the distribution of values for a row of the Stern sequence in the diatomic array is not an easy problem. As a means of trying to understand the distribution of the Stern sequence, we consider a potentially similar sequence, the binomial coefficients. The initial idea that these two sequences might be comparable in behavior of distribution came from the sum of values in a row. For example, the sum of values in the n-th row of the diatomic array is 3^n, and the sum of the binomial coefficients $\binom{n}{k}$ for a fixed n is 2^n. With this motivation of understanding distribution of the Stern sequence better, we investigated the distribution of values of the binomial coefficients.

4.2 The Distribution of Values for the Binomial Coefficients

As a means of understanding the distribution of the values of the binomial coefficients, we compare them to the average value. First note that the average value of the binomial coefficients is $2^n/(n+1)$.

For a fixed n, we define

$$F(\lambda, n) := \frac{\#\left\{0 \leq k \leq n : \binom{n}{k} \geq \lambda \frac{2^n}{n+1}\right\}}{n+1}$$

to be the counting function for the number of binomial coefficients which are larger than λ times the average value.

Remark 5. A probabilistic interpretation of this would be finding at how many points k, with $k = 0, 1, \ldots, n$, does the probability mass function $f_k(n) := \binom{n}{k}2^{-n}$ lie above $\lambda/(n+1)$. However, we will proceed from a number theoretic standpoint.

Then for various values of n, we compute $F(\lambda, n)$ to see if there is a limiting function as n goes to infinity. Figure 4.1 compares the values of $F(\lambda, 2^8)$, $F(\lambda, 2^9)$, and $F(\lambda, 2^{10})$. The curve of $F(\lambda, 2^8)$ is given in blue

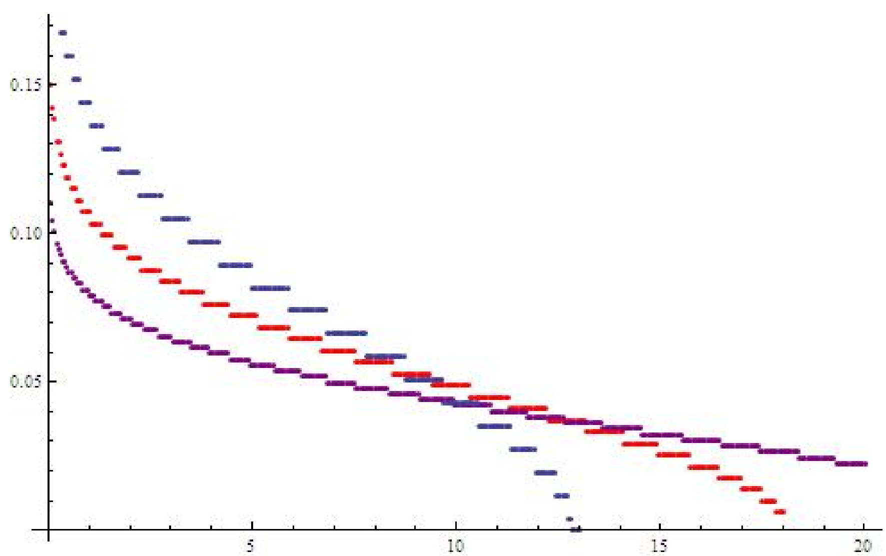


Figure 4.1: Comparing $F(\lambda, 2^8)$, $F(\lambda, 2^9)$, and $F(\lambda, 2^{10})$

and is on top for λ close to 0. The curve of $F(\lambda, 2^9)$ is given in red and is the middle curve. The curve for $F(\lambda, 2^{10})$ is given in purple and is on bottom for λ close to 0. The data suggests that a nontrivial limiting function does not exist, possibly because the comparing function grows too fast as compared to the binomial coefficients. It is likely $F(\lambda, n)$ converges to $F(\lambda) = 0$ for $\lambda > 0$, with $F(0) = 1$.

4.3 A Variation for the Binomial Coefficients

We now vary the problem in perhaps an unexpected way. Instead of allowing the variable λ to be a multiplier, we want the counting function to converge to a limiting function, so we raise the average value to λ instead. We now define the counting function to be

$$G(\lambda, n) := \frac{\#\left\{0 \le k \le n : \binom{n}{k} \ge \left(\frac{2^n}{n+1}\right)^{\lambda}\right\}}{n+1}.$$

We will also denote the limit, whose existence we shall establish later, by

$$G(\lambda) := \lim_{n\to\infty} G(\lambda, n).$$

Computing $G(\lambda, n)$ for various n, as seen in Figure 4.2, we see that a limiting function does seem to exist. The sequence of functions seems to converge to $G(\lambda)$ fairly quickly; the error is roughly 0.0125 for $G(\lambda, 2^{10})$, 0.00556 for $G(\lambda, 2^{11})$, and 0.00312 for $G(\lambda, 2^{13})$. This data then leads us to the following theorem.

Theorem 4.3.1. *The limit $G(\lambda)$ exists, and satisfies the relation*

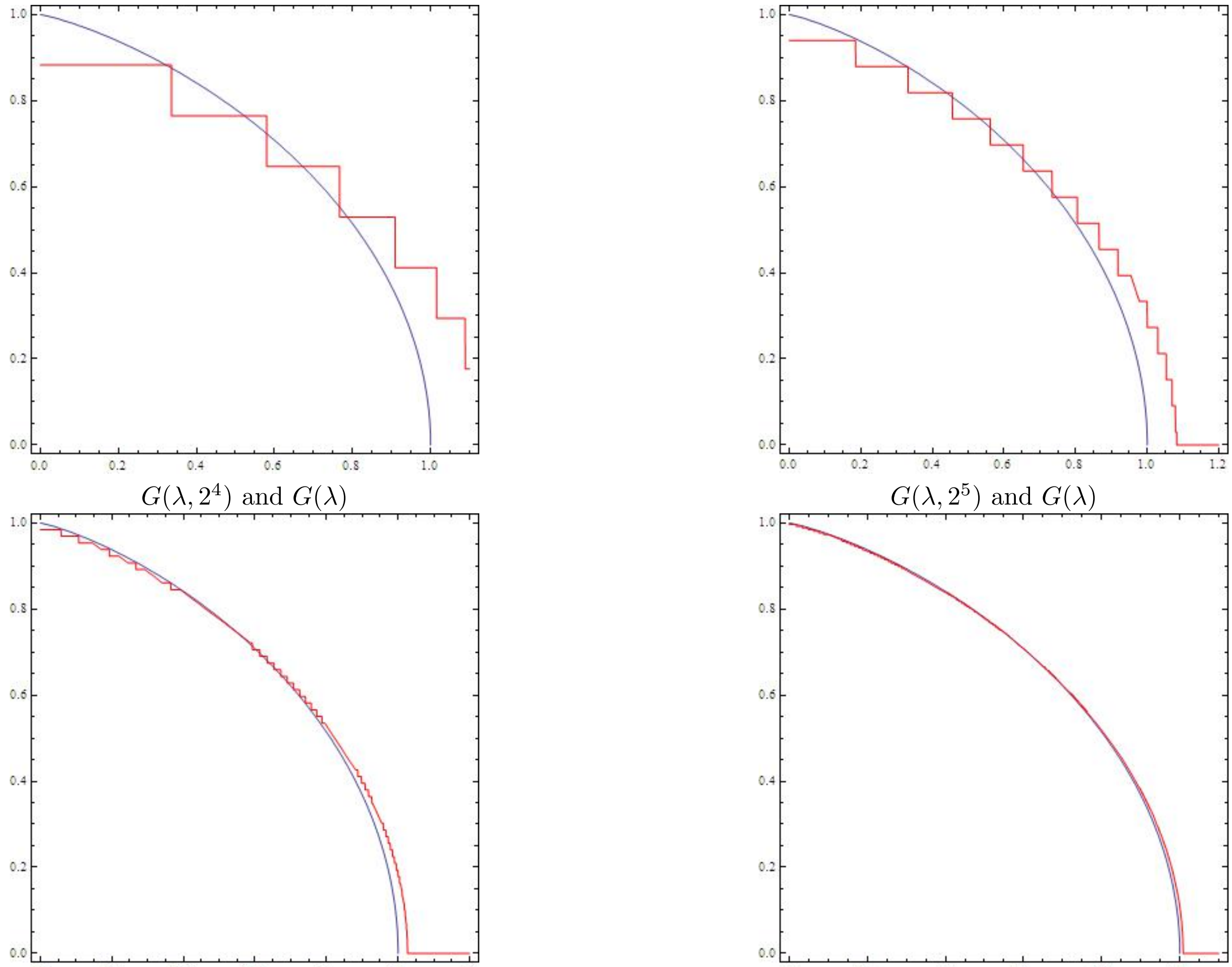
$$1 - \frac{(1 + G(\lambda))\ln(1 + G(\lambda)) + (1 - G(\lambda))\ln(1 - G(\lambda))}{2\ln 2} = \lambda. \tag{4.1}$$

While refinements of the asymptotics of the binomial coefficients can be found in [35], basic asymptotics and Stirling's Formula are sufficient to prove the result.

Proof. The larger values for the binomial coefficient occur in the middle, at approximately $n/2$. Our main term will come from $\sqrt{n} \le k \le n - \sqrt{n}$, but since the binomial coefficients are also symmetric, we will consider only the second half, or more specifically $\frac{n}{2} \le k \le n - \sqrt{n}$, for our initial estimates. For the tails we only need consider k in the range $0 \le k < \sqrt{n}$, again, because of symmetry.

We first derive estimates for the main term. If we let $k = \frac{n}{2} + m$, we have $0 \le m \le \frac{n}{2} - \sqrt{n}$. Then clearly

$$\sqrt{n/2} \le \sqrt{n/2 + m} \le \sqrt{n - \sqrt{n}} \le \sqrt{n},$$



$G(\lambda, 2^4)$ and $G(\lambda)$

$G(\lambda, 2^5)$ and $G(\lambda)$

$G(\lambda, 2^7)$ and $G(\lambda)$

$G(\lambda, 2^9)$ and $G(\lambda)$

Figure 4.2: $G(\lambda, 2^4)$, $G(\lambda, 2^5)$, $G(\lambda, 2^7)$, $G(\lambda, 2^9)$ compared with $G(\lambda)$

so that $(n/2 + m)^{-1/2} = O(n^{-1/2})$. Similarly we have

$$\sqrt[4]{n} \leq \sqrt{n/2 - m} \leq \sqrt{n/2},$$

so that $(n/2 - m)^{-1/2} = O(n^{-1/4})$. Also note that $(1 + O(n^{-1/2}))(1 + O(n^{-1/4})) = (1 + O(n^{-1/4}))$.

Using Stirling's Formula three times, we have

$$\begin{aligned} \binom{n}{\frac{n}{2} + m} &= \frac{n!}{(n/2 - m)!(n/2 + m)!} \\ &= \frac{n^n \sqrt{n}}{\sqrt{2\pi}(n/2 + m)^{n/2+m+1/2}(n/2 - m)^{n/2-m+1/2}} \left(1 + O(n^{-1/4})\right). \end{aligned}$$

Now pick m such that $\binom{n}{n/2+m} \geq (2^n/(n+1))^\lambda$ but $\binom{n}{n/2+m+1} < (2^n/(n+1))^\lambda$. Since logarithmic functions are monotonic, we have

$$\#\left\{0 \leq k \leq n : \binom{n}{k} \geq \left(\frac{2^n}{n+1}\right)^\lambda\right\} = \#\left\{0 \leq k \leq n : \ln\binom{n}{k} \geq \ln\left(\frac{2^n}{n+1}\right)^\lambda\right\},$$

so we can consider the inequality $\ln\binom{n}{n/2+m} \geq \ln(2^n/(n+1))^\lambda$ and simplify. So we have

$$\begin{aligned}\lambda n \ln 2 - \lambda \ln(n+1) &\leq n\ln n + \frac{1}{2}\ln n - \frac{1}{2}\ln(2\pi) - \left(\frac{n}{2}+m\right)\ln\left(\frac{n}{2}+m\right) - \left(\frac{n}{2}-m\right)\ln\left(\frac{n}{2}-m\right) \\ &\quad - \frac{1}{2}\ln\left(\frac{n}{2}+m\right) - \frac{1}{2}\ln\left(\frac{n}{2}-m\right) + \ln(1+O(n^{-1/4})) \\ &\leq n\ln n + \left(\frac{n}{2}+m\right)\ln\left(\frac{n}{2}+m\right) - \left(\frac{n}{2}-m\right)\ln\left(\frac{n}{2}-m\right) \\ &\quad + O(\ln n) + O(n^{-1/4}).\end{aligned}$$

This implies

$$\lambda n \ln 2 + O(\ln n) \leq n\ln n + \left(\frac{n}{2}+m\right)\ln\left(\frac{n}{2}+m\right) - \left(\frac{n}{2}-m\right)\ln\left(\frac{n}{2}-m\right) + O(\ln n) + O(n^{-1/4}).$$

After dividing by n, and rearranging, we have

$$\lambda \ln 2 \leq \ln n - \left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{n}{2}+m\right) - \left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{n}{2}-m\right) + O\left(\frac{\ln n}{n}\right) + O(n^{-5/4}).$$

The right hand side simplifies to

$$\begin{aligned}&\ln n - \left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{n}{2}+m\right) - \left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{n}{2}-m\right) + O(n^{-5/4}) \\ &\quad = \ln n - \left(\frac{1}{2}+\frac{m}{n}\right)\ln n - \left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{1}{2}+\frac{m}{n}\right) - \left(\frac{1}{2}-\frac{m}{n}\right)\ln n \\ &\qquad - \left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{1}{2}-\frac{m}{n}\right) + O(n^{-5/4}) \\ &\quad = -\left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{1}{2}+\frac{m}{n}\right) - \left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{1}{2}-\frac{m}{n}\right) + O(n^{-5/4}).\end{aligned}$$

We then have the inequality

$$\left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{1}{2}+\frac{m}{n}\right) + \left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{1}{2}-\frac{m}{n}\right) + O(n^{-5/4}) \geq \lambda \ln 2. \tag{4.2}$$

Because of our choice of m, (4.2) implies

$$\left(\frac{1}{2}+\frac{m+1}{n}\right)\ln\left(\frac{1}{2}+\frac{m+1}{n}\right)+\left(\frac{1}{2}-\frac{m+1}{n}\right)\ln\left(\frac{1}{2}-\frac{m+1}{n}\right)+O(n^{-5/4})\leq -\lambda\ln 2. \tag{4.3}$$

Simplifying the left side of the inequality (4.3), we have

$$\left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{1}{2}+\frac{m}{n}\right)+\left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{1}{2}-\frac{m}{n}\right)+O\left(\frac{1}{n}\right)\leq -\lambda\ln 2,$$

and this implies

$$\left(\frac{1}{2}+\frac{m}{n}\right)\ln\left(\frac{1}{2}+\frac{m}{n}\right)+\left(\frac{1}{2}-\frac{m}{n}\right)\ln\left(\frac{1}{2}-\frac{m}{n}\right)=-\lambda\ln 2+O\left(\frac{1}{n}\right). \tag{4.4}$$

Now, let $g(t) := (\frac{1}{2}+t)\ln(\frac{1}{2}+t)+(\frac{1}{2}-t)\ln(\frac{1}{2}-t)$. If $t=\frac{m}{n}$, then $0\leq t\leq \frac{1}{2}-n^{-1/2}$ and $g:[0,\frac{1}{2}-n^{-1/2}]\to\mathbb{R}$. Otherwise, g is well defined on the interval $[0,\frac{1}{2}]$, with $g(0)=-\ln 2$ and $g(\frac{1}{2})=0$. Looking at $g'(t)=\ln(\frac{1}{2}+t)-\ln(\frac{1}{2}-t)$, we see $g'(t)>0$ for $t>0$. This means g is a strictly increasing function on the interval $[0,\frac{1}{2}]$, and there exists a unique $t_\lambda\in(0,\frac{1}{2}-n^{-1/2})$ such that $g(t_\lambda)=-\lambda\ln 2$. So if $g(m/n)=-\lambda\ln 2+O(n^{-1/2})=g(t_\lambda)+O(n^{-1/2})$, then this implies $m/n=t_\lambda+O(n^{-1/2})$, so that $m=nt_\lambda+O_\lambda(\sqrt{n})$.

Now, we have

$$\begin{aligned} G(\lambda,n) &= \frac{2\#\left\{0\leq m\leq \frac{n}{2}-\sqrt{n}:\binom{n}{n/2+m}\geq\left(\frac{2^n}{n+1}\right)^\lambda\right\}}{n+1}+\frac{2\#\left\{0\leq k\leq\sqrt{n}:\binom{n}{k}\geq\left(\frac{2^n}{n+1}\right)^\lambda\right\}}{n+1}\\ &= \frac{2t_\lambda n+O(n^{1/2})}{n+1}+\frac{O(\sqrt{n})}{n+1}\\ &= \frac{2t_\lambda n}{n+1}+O(n^{-1/2}). \end{aligned}$$

Then taking the limit, we find

$$\lim_{n\to\infty}G(\lambda,n)=\lim_{n\to\infty}\frac{2t_\lambda n}{n+1}+O(n^{-1/2})=2t_\lambda,$$

so that the limit $G(\lambda)$ exists.

All that is left is to show $G(\lambda)$ satisfies (4.1). We have

$$\begin{aligned}
1 - \frac{1}{2\ln 2}(1+G(\lambda))\ln(1+G(\lambda)) &+ (1-G(\lambda))\ln(1-G(\lambda)) \\
&= 1 - \frac{(1+2t_\lambda)\ln(1+2t_\lambda)+(1-2t_\lambda)\ln(1-2t_\lambda)}{2\ln 2} \\
&= 1 - \frac{2(\frac{1}{2}+t_\lambda)(\ln 2+\ln(\frac{1}{2}+t_\lambda))+2(\frac{1}{2}-t_\lambda)(\ln 2+\ln(\frac{1}{2}-t_\lambda))}{2\ln 2} \\
&= 1 - \frac{2g(t_\lambda)+2\ln 2}{2\ln 2} \\
&= 1 - \frac{-2\lambda\ln 2+2\ln 2}{2\ln 2} \\
&= 1-(1-\lambda) \\
&= \lambda. \qquad \square
\end{aligned}$$

Remark 6. There are asymptotic estimates for the binomial coefficients which involve the binary entropy function (see [15]). This helps to explain why the function g in the proof of Theorem 4.3.1, as well as the relation which $G(\lambda)$ satisfies, is reminiscent of the binary entropy function.

4.4 The Distribution of Values for the Stern sequence

There are a variety of ways to view the distribution of values for the Stern sequence. Figure 4.3 shows the values of the Stern sequence in the 10th and 14th rows of the diatomic array. The values are very symmetric

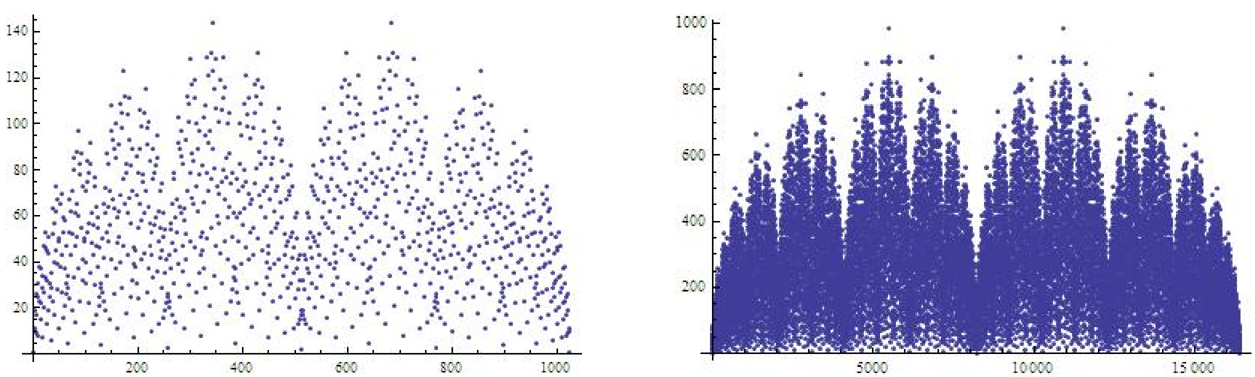


Figure 4.3: Values for the 10th and 14th rows of $s(n)$

and fractal-like. Another way to look at the values, is to take a row and order it. In Figure 4.4, we took the 14th row of the Stern sequence, arranged the values in increasing order, and then normalized the values by dividing by the largest value F_{16}. We can also consider the frequency of each value. Figure 4.5 shows the frequency of each value that occurs in the 14th row of the Stern sequence. The graph looks like a filled

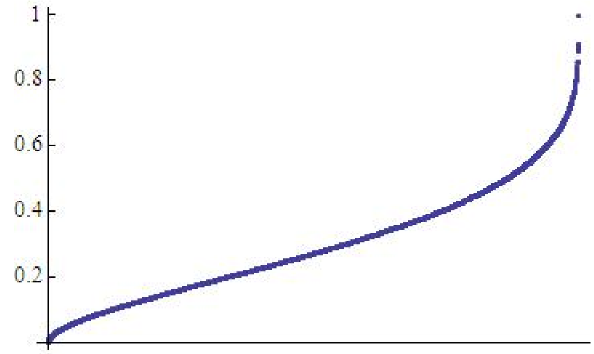


Figure 4.4: Normalized values in the 14th row

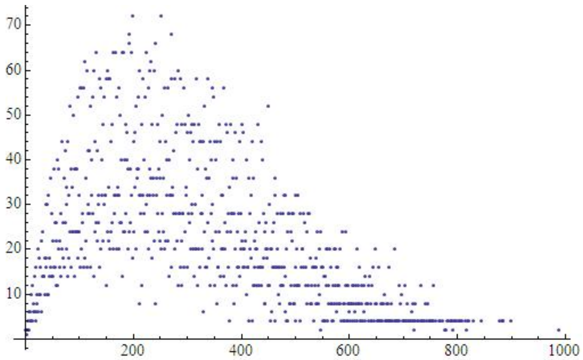


Figure 4.5: Frequency of values in 14th row

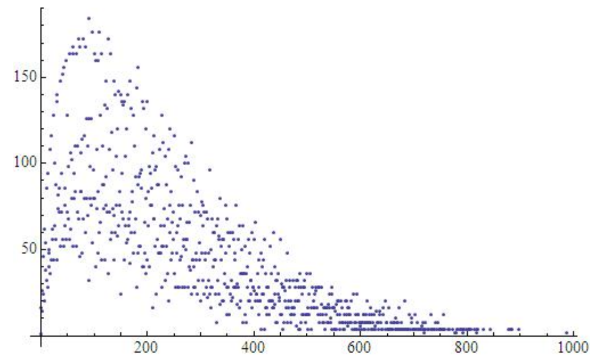


Figure 4.6: Frequency of values up to 14th row

in normal distribution. If we consider the frequency of values cumulatively up to the 14th row, as given in Figure 4.6, the graph looks like a skewed normal distribution.

We can also analyze the distribution of values by comparing them to the average value. Recall that Stern [34] showed the sum of the values in a row of the diatomic array is a power of 3, so that

$$\sum_{n=2^r}^{2^{r+1}-1} s(n) = 3^r. \tag{4.5}$$

This implies the average value of the r-th row is roughly $(3/2)^r$. For fixed N, this implies

$$\frac{1}{N}\sum_{n=0}^{N} s(n) \asymp N^{\beta-1}, \tag{4.6}$$

where $\beta = \log_2 3$.

For the distribution of values of the Stern sequence, we then count the number of terms in a row of the diatomic array which are larger than the average value. We define the counting function

$$H(\lambda, N) := \frac{\#\left\{2^N \le n < 2^{N+1} : s(n) \ge \lambda\left(\frac{3^N}{2^N}\right)\right\}}{2^N}.$$

The data in Figure 4.7 suggests $H(\lambda, N)$ converges to a smooth function, but it is not clear if it actually does. Overall, the general shape of the graphs looks like e^{-ax^2-bx}, but the data does not stay close to the

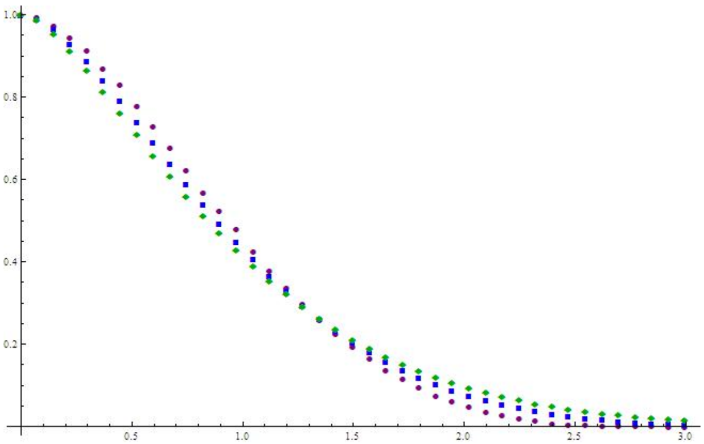


Figure 4.7: $H(\lambda, 2^{12})$ (circles), $H(\lambda, 2^{17})$ (squares), and $H(\lambda, 2^{22})$ (diamonds)

curve. We hope to understand the nature of this limiting distribution in the future.

As it turns out, the binomial coefficients are not similar enough to gain any information for the Stern sequence. The next step after this problem will be to understand the distribution of gaps of the Stern sequence.

4.5 The Distribution of Gaps

There are numerous ways we could consider the gaps of the Stern sequence. First, we can simply compute $s(n+1) - s(n)$. Figure 4.8 plots the normalized difference for the 14th row. Figure 4.8 looks like the plot of $s(n)$ with also a reflection over the x-axis. If $n = 2\ell$, we have

$$s(n+1) - s(n) = s(2\ell+1) - s(2\ell) = s(\ell+1).$$

If $n = 2\ell + 1$, then we have

$$s(n+1) - s(n) = s(2\ell+2) - s(2\ell+1) = -s(\ell).$$

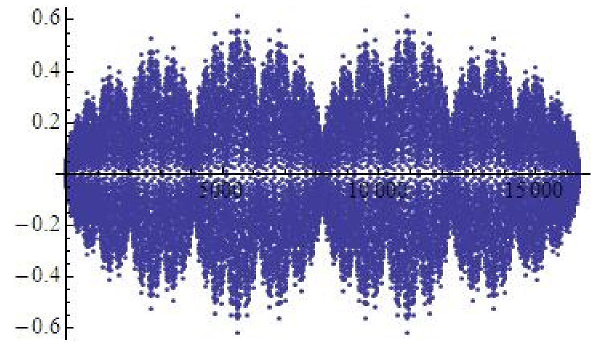


Figure 4.8: $s(n+1) - s(n)$ for the 14th row

From this perspective, the distribution of gaps is the same problem as the distribution of values.

Another way to approach this problem is to order a row from largest to smallest and then take the difference of consecutive terms. Since values occur at least twice in a row, we would have a lot of zeros. If we eliminate repeated values when taking the difference, then we have a type of step function. Figures 4.9

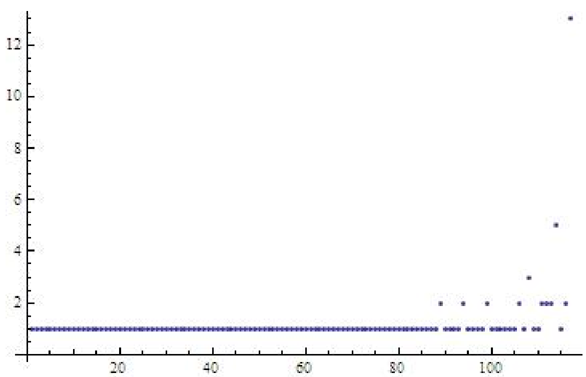


Figure 4.9: Length of gaps for the 10th row

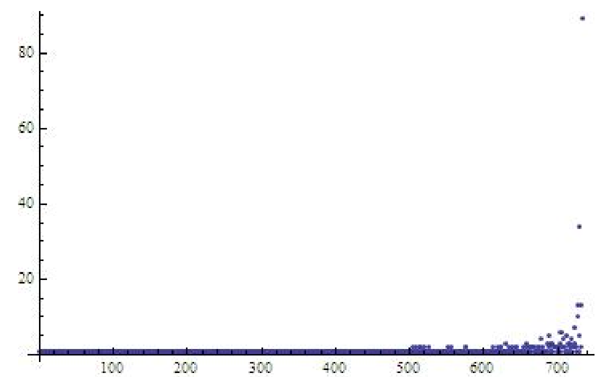


Figure 4.10: Length of gaps for the 14th row

and 4.10 give the differences of terms for the 10th row and the 14th row, after they have been sorted and repeated elements deleted.

In Chapter 2, we discussed the largest three values in a row of the diatomic array, and we can use this information here. First, we normalize the data by dividing by the largest possible value F_{r+2}. By Theorems 2.2.2 and 2.3.2, we have the biggest gap is F_{r-3}/F_{r+2}, and the next gap is F_{r-7}/F_{r+2}. If the conjecture $L_k(r) - L_{k-1}(r) = F_{r-(4k-5)}$ is true, then the distribution of gaps is $F_{r-(4k-5)}/F_{r+2}$ for $2 \leq k \leq (r+5)/4$, with the smallest value being $1/F_{r+2}$. The largest gaps are fixed, regardless of the row. The smallest gaps then approach 0, for the later rows.

This would mean the distribution of the gaps is not uniformly distributed. If Conjecture 2.3.4 is true, then we have

Conjecture 4.5.1. *The normalized distribution of gaps for the Stern sequence is* $\phi^{-(4k-3)}$ *for* $2 \leq k \leq (r+5)/4$.

Chapter 5

Properties of $w(n)$

5.1 Basic Properties

Recall the definition

$$w(n) := \frac{1}{2}s(3n),$$

and that $w(n)$ is also defined independently of $s(n)$ by the recurrences

$$w(2n) = w(n), \quad w(8n \pm 1) = w(4n \pm 1) + 2w(n),$$

$$w(8n \pm 3) = w(4n \pm 1) + w(2n \pm 1) - w(n), \text{ for } n \geq 1,$$

$$\text{with } w(0) = 0,\ w(1) = 1, \text{ and } w(3) = 2.$$

These recurrences were proved in Chapter 1. We repeat the table of values given in Chapter 1. Table 5.1 gives a comparison of the first 64 values of $s(n)$ and $w(n)$. Examining Table 5.1, for small n we see that

Table 5.1: Values for $s(n)$ and $w(n)$

n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$
1	1	1	17	5	6	33	6	8	49	9	13
2	1	1	18	4	4	34	5	6	50	7	9
3	2	2	19	7	5	35	9	9	51	12	12
4	1	1	20	3	2	36	4	4	52	5	5
5	3	2	21	8	3	37	11	7	53	13	8
6	2	2	22	5	3	38	7	5	54	8	7
7	3	4	23	7	7	39	10	9	55	11	15
8	1	1	24	2	2	40	3	2	56	3	4
9	4	4	25	7	9	41	11	7	57	10	17
10	3	2	26	5	5	42	8	3	58	7	9
11	5	3	27	8	7	43	13	4	59	11	11
12	2	2	28	3	4	44	5	3	60	4	6
13	5	5	29	7	9	45	12	8	61	9	13
14	3	4	30	4	6	46	7	7	62	5	8
15	4	6	31	5	8	47	9	11	63	6	10
16	1	1	32	1	1	48	2	2	64	1	1

$w(2n) < w(2n+1)$ and $w(2n+2) \leq w(2n+1)$.

Theorem 5.1.1. *For all $n \geq 0$, we have*

$$w(2n) < w(2n+1) \quad \textit{and} \quad w(2n+2) \leq w(2n+1),$$

with equality when $n = \frac{2}{3}(4^{t-1} - 1)$.

Proof. Note that $w(2n) < w(2n+1)$ is equivalent to showing

$$s(6n+3) = s(3n+1) + s(3n+2) > s(3n). \tag{5.1}$$

Let $3n+1 = 2^r m$, where m is odd. Then (5.1) becomes

$$s(2^r m - 1) < s(2^r m) + s(2^r m + 1).$$

Using (1.3), we have

$$s(2^r m) + s(2^r m + 1) = s(m) + s(2^r - 1)s(m) + s(m+1)$$

and

$$s(2^r m - 1) = s(2^r - 1)s(m) + s(m-1).$$

Thus, showing (5.1) holds is equivalent to showing $s(m-1) < s(m) + s(m+1)$. Since m is odd, and using (1.5), we have $s(m) > s(m-1)$, so that (5.1) holds.

Using the same approach for the other inequality, we have $w(2n+2) \leq w(2n+1)$ is equivalent to showing $s(3n+3) \leq s(3n+1) + s(3n+2)$. Writing $3n+2 = 2^r m$ where m is odd, we have $s(2^r m + 1) \leq s(2^r m - 1) + s(2^r m)$. Again, using (1.3), $s(2^r m + 1)$ reduces to $s(2^r - 1)s(m) + s(m+1)$, and $s(2^r m - 1) + s(2^r m)$ reduces to $s(2^r - 1)s(m) + s(m-1) + s(m)$. Then we need to show $s(m+1) \leq s(m-1) + s(m)$. Since m is odd, let $m = 2t+1$, so we have $s(m+1) = s(t+1) \leq 2s(t) + s(t+1)$, which is true. Note we have equality if $n = 2(4^t - 1)/3$. Using (1.4), we have

$$\begin{aligned} w\left(2 \cdot \frac{2(4^t - 1)}{3} + 1\right) = w\left(\frac{4^{t+1} - 1}{3}\right) &= \frac{1}{2}s(4^{t+1} - 1) \\ &= \frac{1}{2}s(2^{2t+2} - 1) \\ &= \frac{1}{2}(2t+2) = t+1, \end{aligned}$$

and

$$w\left(2 \cdot \frac{2(4^t-1)}{3}+2\right) = w\left(\frac{2(4^t-1)}{3}+1\right) = \frac{1}{2}s(2\cdot 4^t+1)$$
$$= \frac{1}{2}s(2^{2t+1}+1) = \frac{1}{2}(2t+1+1)$$
$$= t+1. \qquad \square$$

5.2 Symmetry

The symmetry of the Stern sequence turns out to be very useful, and $w(n)$ also inherits the symmetry as well, although the symmetry takes a slightly different form. For the diatomic array, the row in which $s(n)$ would appear is given by $\lfloor \log_2(n) \rfloor$, so that we can expect the row in which $w(n)$ would appear to be $\lfloor \log_2(3n)) \rfloor$. We define the *row number* of $w(n)$ to be given by

$$r(n) := \lfloor \log_2(3n)) \rfloor.$$

Let $r(n) = k$ and define $n' = 2^k - n$. Then $w(n)$ satisfies the symmetry given by $w(n) = w(n')$. To see this, simply note that $2^k < 3n < 2^{k+1}$, implying that $(3n)^* = 3 \cdot 2^k - 3n$. Then we have

$$2w(n) = s(3n) = s((3n)^*) = s(3\cdot 2^k - 3n) = s(3(2^k - n)) = 2w(2^k - n) = 2w(n').$$

If $2^k < 3n < 2^{k+1}$, we have $2^k < 3 \cdot 2^k - 3n < 2^{k+1}$, implying n' also lies in the same row as n, so that $r(n') = k$. It is also worth mentioning the interval of n such that $r(n) = k$ for fixed k, has length roughly $2^k/3$, with midpoint 2^{k-1}. The row in which n is located is symmetric about $2^{r(n)-1}$.

In Chapter 1, we discussed a more subtle type of symmetry for the Stern sequence. If we consider the binary representation for n, we define $\overleftarrow{n}$ to be the reversal of the binary digits of n. There is a four-fold symmetry for the Stern sequence:

$$s(n) = s(n^*) = s(\overleftarrow{n}) = s(\overleftarrow{n^*}).$$

We have seen that $\overleftarrow{n^*} = \overleftarrow{n}^*$. These two symmetries form a nice group with four elements, and these elements will always have the same Stern value. However, there are two special cases where there are only two elements: when n is symmetric in binary (so that ($n = \overleftarrow{n}$ and $n^* = \overleftarrow{n^*}$), and when $\overleftarrow{n} = n^*$ and $n = \overleftarrow{n^*}$.

Does this more subtle symmetry also carry over for $w(n)$? Writing n and $\overleftarrow{n}$ in its binary expansion with

$\epsilon_i \in \{0,1\}$, we find

$$n = \sum_{i=0}^{v} \epsilon_i 2^i \equiv \sum_{i=0}^{v} \epsilon_i (-1)^i \pmod 3,$$

and

$$\overleftarrow{n} = \sum_{i=0}^{v} \epsilon_{v-i} 2^i \equiv \sum_{i=0}^{v} \epsilon_{v-i} (-1)^i \equiv (-1)^v \sum_{i=0}^{v} \epsilon_{v-i} (-1)^{v-i} \pmod 3.$$

This means that if 3 divides n, then 3 also divides $\overleftarrow{n}$, hence $w(n/3) = w(\overleftarrow{n}/3)$.

5.3 Combinatorial Interpretation for $w(n)$

The generating function of $s(n)$ is given by

$$\mathcal{S}(x) := x \prod_{j=0}^{\infty} (1 + x^{2^j} + x^{2^{j+1}}).$$

Carlitz [6] remarked this infinite product corresponds to the number of ways to write an integer using powers of 2, with each part appearing at most twice. If we define $b(n)$ as the number of ways to write n as

$$n = \sum_{i=0}^{\infty} \epsilon_i 2^i \quad \text{with} \quad \epsilon_i \in \{0,1,2\}, \tag{5.2}$$

then $s(n+1) = b(n)$. Reznick [28] connected $s(n)$ to this generating function and combinatorial interpretation explicitly. Note that the number of parts, $\sum \epsilon_i$, is congruent to the number of times $\epsilon_i = 1$, modulo 2.

A combinatorial interpretation for $w(n)$ completely independent of $s(n)$ is not easy. For now we consider an interpretation for $w(n)$ in terms of the interpretation for $s(n)$.

First note $s(3n)$ counts the number of ways to write $3n-1$ using powers of two, with each part appearing at most twice. Now, $s(3n)$ is always even. If we think of this in terms of partitions, it means there are an even number of partitions to write $3n-1$ in this way. Then we need to take half of $s(3n)$, since $w(n) = \frac{1}{2}s(3n)$. We now show the number of partitions for $3n-1$ with an even number of parts is equal to the number of partitions with an odd number of parts. This means we can interpret $w(n)$ to be the number of ways to write $3n-1$ using powers of two with each part appearing at most twice, with an even number of parts. Or, we could also interpret it to be the number of ways to write $3n-1$ using powers of two with each part appearing at most twice, with an odd number of parts.

Theorem 5.3.1. *The number of even partitions of $3n+2$ into powers of 2 with each part appearing at most twice is equal to the number of odd partitions of $3n+2$ into powers of 2 with each part appearing at most twice.*

Proof. Consider the generating function

$$\prod_{j=0}^{\infty}(1 + yx^{2^j} + x^{2^{j+1}}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}y^m x^n.$$

The $a_{m,n}$ count the number of ways to write n as $\sum_{i=0}^{\infty} \epsilon_i 2^i$, with $\epsilon_i \in \{0, 1, 2\}$, with precisely m ϵ_i's equal to 1. Let $y = -1$. If a partition of n has an even number of parts, each appearing exactly once, then $a_{m,n}$ will be positive. If the partition has an odd number of parts, each appearing exactly once, then $a_{m,n}$ will be negative. Also, if a part appears 0 or 2 times, this does not change the sign of $a_{m,n}$, so that this generating function counts the number of even partitions and the odd partitions for n. Expanding this generating function, we have

$$\begin{aligned}
\prod_{j=0}^{\infty}(1 - x^{2^j} + x^{2^{j+1}}) &= \prod_{j=0}^{\infty} \frac{(1 - x^{2^j} + x^{2^{j+1}})(1 + x^{2^j} + x^{2^{j+1}})(1 - x^{2^j})}{(1 - x^{3\cdot 2^j})} \\
&= \prod_{j=0}^{\infty} \frac{(1 + x^{2^{j+1}} + x^{2^{j+2}})(1 - x^{2^j})}{(1 - x^{3\cdot 2^j})} \\
&= (1 - x)\prod_{j=0}^{\infty} \frac{(1 + x^{2^{j+1}} + x^{2^{j+2}})(1 - x^{2^{j+1}})}{(1 - x^{3\cdot 2^j})} \\
&= (1 - x)\prod_{j=0}^{\infty} \frac{(1 - x^{3\cdot 2^{j+1}})}{(1 - x^{3\cdot 2^j})} \\
&= \frac{1 - x}{1 - x^3} \\
&= 1 - x + x^3 - x^4 + x^6 - x^7 + \cdots . \qquad (5.3)
\end{aligned}$$

The missing terms are those with powers congruent to 2 modulo 3. Hence, for numbers of the form $3k + 2$, the number of even partitions is equal to the number of odd partitions. □

In the future, we hope to find a bijective proof of this theorem.

5.4 Generating Function

What will the generating function for $w(n)$ be? Will it be similar to the generating function for $s(n)$?

Let

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} w(n)x^n$$

be the generating function for $w(n)$. Recall that for $n \geq 1$, $w(n)$ can be independently defined by the recursions

$$
\begin{aligned}
w(2n) &= w(n) \\
w(8n \pm 1) &= w(4n \pm 1) + 2w(n), && (5.4) \\
w(8n \pm 3) &= w(4n \pm 1) + w(2n \pm 1) - w(n), && (5.5)
\end{aligned}
$$

with $w(0) = 0$, $w(1) = 1$, and $w(3) = 2$. We will use these recurrences to find a recurrence for the generating function.

First, we split up the sum into arithmetic progressions modulo 8. We have

$$
\begin{aligned}
\sum_{n=0}^{\infty} w(n)x^n &= \sum_{n=0}^{\infty} w(8n)x^{8n} + \sum_{n=0}^{\infty} w(8n+1)x^{8n+1} + \cdots + \sum_{n=0}^{\infty} w(8n+6)x^{8n+6} + \sum_{n=0}^{\infty} w(8n+7)x^{8n+7} \\
&= \sum_{n=0}^{\infty}(w(4n)(x^2)^{4n} + w(4n+1)(x^2)^{4n+1} + w(4n+2)(x^2)^{4n+2} + w(4n+3)(x^2)^{4n+3}) \\
&\quad + \sum_{n=0}^{\infty} w(8n+1)x^{8n+1} + \sum_{n=0}^{\infty} w(8n+3)x^{8n+3} + \sum_{n=0}^{\infty} w(8n+5)x^{8n+5} + \sum_{n=0}^{\infty} w(8n+7)x^{8n+7} \\
&= \sum_{n=0}^{\infty} w(n)(x^2)^n + \sum_{n=0}^{\infty} w(8n+1)x^{8n+1} + \cdots + \sum_{n=0}^{\infty} w(8n+7)x^{8n+7}.
\end{aligned}
$$

Then using (5.4) and (5.5), we have

$$
\begin{aligned}
\mathcal{W}(x) &= \mathcal{W}(x^2) + \sum_{n=0}^{\infty}[w(4n+1) + 2w(n)]x^{8n+1} + \sum_{n=0}^{\infty}[w(4n+1) + w(2n+1) - w(n)](x)^{8n+3} \\
&\quad + \sum_{n=1}^{\infty}[w(4n-1) + w(2n-1) - w(n)](x)^{8n-3} + \sum_{n=1}^{\infty}[w(4n-1) + 2w(n)]x^{8n-1} \\
&= \mathcal{W}(x^2) + 2(x + x^{-1})\mathcal{W}(x^8) + (x + x^{-1}) \sum_{n=0}^{\infty} w(4n+1)(x^2)^{4n+1} \\
&\quad + (x + x^{-1}) \sum_{n=0}^{\infty} w(4n+3)(x^2)^{4n+3} + (x + x^{-1}) \sum_{n=0}^{\infty} w(4n+2)(x^2)^{4n+2} \\
&\quad - (x^3 + x^{-3}) \sum_{n=0}^{\infty} w(n)(x^8)^n \\
&= \mathcal{W}(x^2) - (x^3 - x - x^{-1} + x^{-3})\mathcal{W}(x^8) + (x + x^{-1}) \sum_{n=0}^{\infty} w(4n)(x^2)^{4n} \\
&\quad + (x + x^{-1}) \sum_{n=0}^{\infty}[w(4n+1)(x^2)^{4n+1} + w(4n+3)(x^2)^{4n+3} + w(4n+2)(x^2)^{4n+2}] \\
&= \mathcal{W}(x^2) - (x^3 - x - x^{-1} + x^{-3})\mathcal{W}(x^8) + (x + x^{-1})\mathcal{W}(x^2) \\
&= (x + 1 + x^{-1})\mathcal{W}(x^2) - (x^3 - x - x^{-1} + x^{-3})\mathcal{W}(x^8).
\end{aligned}
$$

Since $w(0) = s(0) = 0$, we deduce $\mathcal{W}(0) = 0$, and thus we can write $\mathcal{W}(x) = xT(x)$. Rewriting $\mathcal{W}(x) = xT(x)$ in the last line above, we find

$$\mathcal{W}(x) = xT(x) = x^2(x^{-1} + 1 + x)T(x^2) - x^8(x^{-3} - x^{-1} - x + x^3)T(x^8), \tag{5.6}$$

which implies

$$T(x) = (1 + x + x^2)T(x^2) - x^4(1 - x^2 - x^4 + x^6)T(x^8).$$

The first expression on the right-hand side satisfies the same recurrence relation which is iterated to obtain the generating function for the Stern sequence. However, the second expression on the right-hand side introduces a perturbation to the recurrence relation for the Stern generating function, making it difficult to compute a nice product representation for the generating function of $w(n)$.

Another approach employs the third root of unity, ω. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f(x) + f(\omega x) + f(\omega^2 x) = 3 \sum_{t=0}^{\infty} a_{3t} x^{3t}.$$

This implies

$$\frac{1}{6}(\mathcal{S}(x) + \mathcal{S}(\omega x) + \mathcal{S}(\omega^2 x)) = \sum_{t=0}^{\infty} \frac{1}{2} s(3t) x^{3t},$$

so that

$$\mathcal{W}(x) = \frac{1}{6}(\mathcal{S}(x^{1/3}) + \mathcal{S}(\omega x^{1/3}) + \mathcal{S}(\omega^2 x^{1/3})).$$

However, this formula for the generating function, while closed, is not very useful. We hope to find an even better form for the generating function in the future.

Chapter 6

Maximum Values for $w(n)$

The maximum value in a row of the Stern sequence is a Fibonacci number. What will the maximum value for $w(n)$ be?

6.1 Maximum Values for $w(n)$

Since $w(n)$ is derived from the Stern sequence, we expect to find the maximum values for each row of $w(n)$ by knowing the largest even number in a row for the Stern sequence. More specifically, we only need consider the three largest values in a row. Since $L_1(r)$, $L_2(r)$, and $L_3(r)$ each satisfy a Fibonacci recurrence which is not all even, every third term in the sequence is even. This simply comes from the fact $odd + even = odd$, $even + odd = odd$, and $odd + odd = even$. So the maximum of $w(n)$ in a row occurs precisely when one of $L_1(r)$, $L_2(r)$,or $L_3(r)$ is even and then it is divided by two. Please note Table 6.1 gives a comparison of the three largest values for $s(n)$ for easy reference. The boxed even values in Table 6.1 are divided by 2 to yield the M_k's in Table 6.2.

Let M_k be the maximum of $w(n)$ in the k-th row. The maximum values for $w(n)$ and where they occur are given in Table 6.2. There is a pattern for M_k and where they occur, based on every third row.

The Fibonacci numbers F_r are even when r is a multiple of 3, which implies $L_1(r) = F_{r+2}$ will be even when $r \equiv 1 \pmod 3$. Now, $L_2(r) = F_{r+2} - F_{r-3}$, and this will be even only when $r \equiv 2 \pmod 3$. If $r \equiv 1 \pmod 3$, then F_{r+2} is even and F_{r-3} is odd, so that their sum is odd. If $r \equiv 0 \pmod 3$, then F_{r+2} is odd, and F_{r-3} would be even. If $r \equiv 2 \pmod 3$, then both are odd, so that $F_{r+2} - F_{r-3}$ is even. We have $L_3(r) = F_{r+2} - 3F_{r-5}$ is even when $r \equiv 0 \pmod 3$.

However, it is important to note the row numbers for $w(n)$ are shifted up one relative to the row numbers for $s(n)$. To avoid confusion in this section, we will denote the row numbers for $s(n)$ with r and the row numbers for $w(n)$ with k (with $k = r + 1$). We then have the following theorem.

Table 6.1: Comparison of the Three Largest Values of $s(n)$ in rows

row r	$L_1(r)$	$L_2(r)$	$L_3(r)$
1	$\boxed{2}$	1	N/A
2	3	$\boxed{2}$	1
3	5	$\boxed{4}$	3
4	$\boxed{8}$	7	5
5	13	$\boxed{12}$	11
6	21	19	$\boxed{18}$
7	$\boxed{34}$	31	30
8	55	$\boxed{50}$	49
9	89	81	$\boxed{80}$
10	$\boxed{144}$	131	129
11	233	$\boxed{212}$	209
12	377	343	$\boxed{338}$

Table 6.2: Maxima of $w(n)$ in rows

row	n	M_k	row	n	M_k
2	1	1	13	1817, 1849, 2247, 2279	169
3	2	1	14	3641, 4551	305
4	3, 5	2	15	7281, 7737, 8647, 9103	449
5	7,9	4	16	14567, 14791, 17977, 18201	716
6	15, 17	6	17	29127, 36409	1292
7	25, 29, 35, 39	9	18	58255, 61895, 69177, 72817	1902
8	57, 71	17	19	116505, 118329, 143815, 145639	3033
9	113, 121, 135, 143	25	20	233017, 291271	5473
10	231, 281	40	21	466033, 495161, 533415, 582543	8057
11	455, 569	72	22	932071, 946631, 1150521, 1165081	12848
12	911, 967, 1081, 1137	106	23	1864135, 2330169	23184

Theorem 6.1.1. *For $k \geq 5$, the maximum value of $w(n)$ in the k-th row is*

$$\begin{aligned} \frac{1}{2}F_{k+1} - \frac{1}{2}F_{k-4} &\quad \textit{when } k \equiv 0 \mod 3, \\ \frac{1}{2}F_{k+1} - \frac{3}{2}F_{k-6} &\quad \textit{when } k \equiv 1 \mod 3, \\ \frac{1}{2}F_{k+1} &\quad \textit{when } k \equiv 2 \mod 3. \end{aligned}$$

Proof. This is a consequence of Theorems 2.1.2, 2.2.2, and 2.3.2. Note when the three largest values for $s(n)$ begin their Fibonacci recurrence (starting at their initial values): $L_1(r)$ for all r, $L_2(r)$ for $r \geq 4$ ($k \geq 5$), and $L_3(r)$ for $r \geq 8$ ($k \geq 9$). However, in the $r = 6$-th row ($k = 7$-th), L_3 is 18 and the largest even value among L_1, L_2, and L_3. We also see that $\frac{1}{2}(F_8 - 3F_1) = \frac{1}{2}(21 - 3) = \frac{1}{2} \cdot 18 = 9$, so the last condition is also true for $k = 7$. This completes the proof. □

Remark 7. Since $2|s(n)$ if and only if $3|n$, the maximum of $w(n)$ will occur when 3 divides the values which produce L_1, L_2, and L_3. This information is organized below in Table 6.3. Also note, every third row M_k will only occur twice because L_1 only appears twice in a row. In the other two rows, the M_k occur four times, since L_2 and L_3 occur four times in a row of the diatomic array.

Table 6.3: $n_1(k)$ such that $w(n_1(k)) = M_k$

k-th row	$n_1(k)$	$n_1(k)'$
$k = 4$	3	5
$k \equiv 2 \mod 3$	$\frac{1}{9}(4 \cdot 2^{k-1} - (-1)^{k-1})$	$\frac{1}{9}(5 \cdot 2^{k-1} + (-1)^{k-1})$
$k \equiv 0 \mod 3$	$\frac{1}{9}(16 \cdot 2^{k-3} - 7(-1)^{k-1})$	$\frac{1}{9}(20 \cdot 2^{k-3} + 7(-1)^{k-1})$
	$\frac{1}{9}(17 \cdot 2^{k-3} - (-1)^{k-1})$	$\frac{1}{9}(19 \cdot 2^{k-3} + (-1)^{k-1})$
$k \equiv 1 \mod 3$	$\frac{1}{9}(64 \cdot 2^{k-5} - 31(-1)^{k-1})$	$\frac{1}{9}(80 \cdot 2^{k-5} + 31(-1)^{k-1})$
	$\frac{1}{9}(65 \cdot 2^{k-5} + (-1)^{k-1})$	$\frac{1}{9}(79 \cdot 2^{k-5} - (-1)^{k-1})$

6.2 Generating Function

Computing the generating functions for M_{3j+1}, M_{3j+2}, and M_{3j+3}, we have

$$f_1(x) = 2x + 9x^2 + 40x^3 + 169x^4 + 716x^5 + \cdots , \tag{6.1}$$

$$f_2(x) = 1 + 4x + 17x^2 + 72x^3 + 305x^4 + 1292x^5 + \cdots , \tag{6.2}$$

$$f_3(x) = 1 + 6x + 25x^2 + 106x^3 + 449x^4 + 1902x^5 + \cdots . \tag{6.3}$$

Then multiplying each of these power series by $1 - 4x - x^2$, we obtain the following closed representations for the generating function:

$$f_1(x) = \frac{2x + x^2 + 2x^3}{1 - 4x - x^2}, \tag{6.4}$$

$$f_2(x) = \frac{1}{1 - 4x - x^2}, \tag{6.5}$$

$$f_3(x) = \frac{1 + 2x}{1 - 4x - x^2}. \tag{6.6}$$

The roots of $1-4x-x^2$ are $2\pm\sqrt{5}$, or ϕ^3 and $\overline{\phi}^3$. Using partial fraction decompositions, we can express the generating functions in terms of power series, and then collect coefficients of x^n to obtain closed formulas for the coefficients in (6.1), (6.2), and (6.3). We have

$$f_1(x) = 7-2x+\left(-\frac{7}{2}+\frac{9}{\sqrt{5}}\right)\sum_{n=0}^{\infty}(2+\sqrt{5})^n x^n+\left(-\frac{7}{2}-\frac{9}{\sqrt{5}}\right)\sum_{n=0}^{\infty}(2-\sqrt{5})^n x^n, \tag{6.7}$$

$$f_2(x) = \left(\frac{1}{2}+\frac{1}{\sqrt{5}}\right)\sum_{n=0}^{\infty}(2+\sqrt{5})^n x^n-\left(\frac{1}{2}-\frac{1}{\sqrt{5}}\right)\sum_{n=0}^{\infty}(2-\sqrt{5})^n x^n, \tag{6.8}$$

$$f_3(x) = \left(\frac{1}{2}+\frac{2}{\sqrt{5}}\right)\sum_{n=0}^{\infty}(2+\sqrt{5})^n x^n+\left(\frac{1}{2}-\frac{2}{\sqrt{5}}\right)\sum_{n=0}^{\infty}(2-\sqrt{5})^n x^n. \tag{6.9}$$

Upon grouping the coefficients for x^n, we recover Theorem 6.1.1. Thus, the M_k satisfy a recurrence relationship, and can be put in terms of ϕ^3. By combining the closed forms of $f_1(x)$, $f_2(x)$, and $f_3(x)$ from (6.4), (6.5), and (6.6), we can obtain a generating function, denoted by $M(x)$, for the maximum values of $w(n)$. First note that $M(x) = xf_1(x^3)+x^2f_2(x^3)+x^3f_3(x^3)$. Then we have

$$M(x) = \frac{x^2+x^3+2x^4+2x^6+x^7+2x^{10}}{1-4x^3-x^6}.$$

Chapter 7

Recurrence Relations

As mentioned in Chapter 1, Reznick [27] found the reduction formula

$$s(2^r n \pm j) = s(n)s(2^r - j) + s(j)s(n \pm 1). \tag{7.1}$$

This formula allows for easier computation of larger values of $s(n)$. Does $w(n)$ have a reduction formula similar to this?

7.1 Recurrences in Arithmetic Progressions

It would be nice to have a reduction formula for $w(2^r n \pm j)$, and so we examine $w(n)$ in arithmetic progressions with this goal in mind. We first compute $w(n)$ in arithmetic progressions modulo 16 and 32, hoping to find a pattern which can easily be generalized. The results are the following:

$$w(16n \pm 1) = w(8n \pm 1) + 2w(n) = w(4n \pm 1) + 4w(n), \tag{7.2}$$

$$w(16n \pm 3) = 2w(4n \pm 1) + w(n), \tag{7.3}$$

$$w(16n \pm 5) = 2w(4n \pm 1) - w(n),$$

$$w(16n \pm 7) = w(8n \pm 3) + 2w(2n \pm 1),$$

and

$$
\begin{aligned}
w(32n \pm 1) &= w(16n \pm 1) + 2w(n) = w(8n \pm 1) + 4w(n) = w(4n \pm 1) + 6w(n), \\
w(32n \pm 3) &= w(16n \pm 3) + 4w(n) = 2w(8n \pm 1) + w(n) = 2w(4n \pm 1) + 5w(n), \\
w(32n \pm 5) &= w(16n \pm 3) + 2w(n) = 2w(8n \pm 1) - w(n) = 2w(4n \pm 1) + 3w(n), \\
w(32n \pm 7) &= 2w(16n \pm 3) - w(n) = 4w(8n \pm 1) - 7w(n) = 4w(4n \pm 1) + w(n), \\
w(32n \pm 9) &= 2w(16n \pm 5) + w(n) = 4w(8n \pm 1) - 9w(n) = 4w(4n \pm 1) - w(n), \\
w(32n \pm 11) &= 2w(16n \pm 7) - 5w(2n \pm 1) = 2w(8n \pm 3) - w(2n \pm 1), \\
w(32n \pm 13) &= 2w(16n \pm 7) - 3w(2n \pm 1) = 2w(8n \pm 3) + w(2n \pm 1), \\
w(32n \pm 15) &= w(16n \pm 7) + 2w(2n \pm 1) = w(8n \pm 3) + 4w(2n \pm 1).
\end{aligned}
$$

Using these recurrences, we can compute the recurrences in arithmetic progressions modulo 64:

$$
\begin{aligned}
w(64n \pm 1) &= w(4n \pm 1) + 8w(n), & w(64n \pm 17) &= 6w(4n \pm 1) - w(n), \\
w(64n \pm 3) &= 2w(4n \pm 1) + 9w(n), & w(64n \pm 19) &= 5w(4n \pm 1) - 2w(n), \\
w(64n \pm 5) &= 2w(4n \pm 1) + 7w(n), & w(64n \pm 21) &= 3w(4n \pm 1) - 2w(n), \\
w(64n \pm 7) &= 4w(4n \pm 1) + 9w(n), & w(64n \pm 23) &= 4w(8n \pm 3) - w(2n \pm 1), \\
w(64n \pm 9) &= 4w(4n \pm 1) + 7w(n), & w(64n \pm 25) &= 4w(8n \pm 3) + w(2n \pm 1), \\
w(64n \pm 11) &= 2w(16n \pm 3) - w(4n \pm 1) & w(64n \pm 27) &= 2w(8n \pm 3) + 3w(2n \pm 1), \\
&= 3w(4n \pm 1) + 2w(n), & w(64n \pm 29) &= 2w(8n \pm 3) + 5w(2n \pm 1) \\
w(64n \pm 13) &= 2w(16n \pm 3) + w(4n \pm 1) & &= 2w(4n \pm 1) + 7w(2n \pm 1) - 2w(n), \\
&= 5w(4n \pm 1) + 2w(n), & w(64n \pm 31) &= w(8n \pm 3) + 6w(2n \pm 1) \\
w(64n \pm 15) &= 6w(4n \pm 1) + w(n), & &= w(4n \pm 1) + 7w(2n \pm 1) - w(n).
\end{aligned}
$$

If we examine $w(2^k n \pm 1)$, we see a nice pattern emerge:

$$
\begin{aligned}
w(8n \pm 1) &= w(4n \pm 1) + 2w(n), & w(32n \pm 1) &= w(16n \pm 1) + 2w(n) \\
w(16n \pm 1) &= w(8n \pm 1) + 2w(n) & &= w(8n \pm 1) + 4w(n) \\
&= w(4n \pm 1) + 4w(n), & &= w(4n \pm 1) + 6w(n).
\end{aligned}
$$

This evidence then suggests the following theorem.

Theorem 7.1.1. *For $1 \le r \le k-2$ and $k \ge 2$, we have*

$$w(2^k n \pm 1) = w(2^{k-r} n \pm 1) + 2rw(n) = w(4n \pm 1) + 2(k-2)w(n).$$

Proof. Using (7.1), we see

$$\begin{aligned} w(2^k n \pm 1) &= \frac{1}{2} s(2^k(3n) \pm 3) = \frac{1}{2}[s(3n)s(2^k - 3) + s(3)s(3n \pm 1)] \\ &= \frac{1}{2}[s(3n)(2k-3) + 2s(3n \pm 1)] = (2k-3)w(n) + s(3n \pm 1). \end{aligned}$$

If we iterate the process with $w(2^{k-r}n \pm 1)$, we have $w(2^{k-r}n \pm 1) = (2(k-r)-3)w(n) + s(3n \pm 1)$. Substituting the first equation into the second and rearranging yields the result. The last part of the above equation is obtained by iteration, or by setting $r = k-2$. □

If we examine the next few entries in the data, we have

$$\begin{aligned} w(16n \pm 3) &= 2w(4n \pm 1) + w(n), \\ w(32n \pm 3) &= 2w(4n \pm 1) + 5w(n), \\ w(64n \pm 3) &= 2w(4n \pm 1) + 9w(n). \end{aligned}$$

This leads us to the next theorem.

Theorem 7.1.2. *For $k > 3$,*

$$w(2^k n \pm 3) = 2w(2^{k-2}n \pm 1) + w(n) = 2w(4n \pm 1) + (4(k-4)+1)w(n).$$

Proof. Using $w(2^5 n \pm 3) = 2w(2^3 n \pm 1) + w(n)$ and replacing n with $2^{k-5}n$, we obtain the first result. By iterating Theorem 7.1.1, we obtain the second. □

Similarly, we arrive at the following theorem.

Theorem 7.1.3. *For $k > 3$,*

$$w(2^k n \pm 5) = 2w(2^{k-2}n \pm 1) - w(n) = 2w(4n \pm 1) + (4(k-4)-1)w(n).$$

Proof. Using $w(2^4n \pm 5) = 2w(2^2n \pm 1) - w(n)$ and replacing n with $2^{k-4}n$, we obtain the first result. Then by iterating Theorem 7.1.1, we have the last equality. □

Another pattern emerged after computing even more recurrence formulas, which led to the following theorem.

Theorem 7.1.4. *For* $k \geq 4$*,*

$$w(2^k n \pm (2^{k-1} - 1)) = w(8n \pm 3) + 2(k-3)w(2n \pm 1).$$

Proof. Note that $w(2^k n \pm (2^{k-1} - 1)) = w(2^{k-1}(2n \pm 1) \mp 1)$, and by substituting n with $2n \pm 1$ in Theorem 7.1.1, we obtain the desired result. □

7.2 Reduction Formulas

										0										
										1										
										1										
									2	1	2									
								2	4	1	4	2								
					3	2	5	4	6	1	6	4	5	2	3					
3	7	2	9	5	7	4	9	6	8	1	8	6	9	4	7	5	9	2	7	3

Table 7.1: Triangular Array for $w(n)$

When writing the table of values for $w(n)$ in a triangular array, we see that the values follow arithmetic progressions down the columns. We can then easily obtain formulas for these by using the recurrence formulas and definition of $s(n)$:

$$\begin{aligned}
&w(2^k) = 1, \\
&w(2^k \pm 1) = 2(k-1), \\
&w(2^k \pm 3) = 4(k-3) + 1, \quad k \geq 4, \\
&w(2^k \pm 5) = 4(k-4) + 3, \quad k \geq 4.
\end{aligned}$$

We can also recover the formula $w(2^k \pm 1) = 2(k-1)$ from Theorem 7.1.1, by letting $n = 1$ in Theorem 7.1.1

and replacing r with $k-2$. Then we find that

$$\begin{aligned} w(2^k \pm 1) &= w(2^{k-(k-2)} \pm 1) + 2(k-2)w(1) \\ &= w(4 \pm 1) + 2(k-2) = 2 + 2(k-2) = 2(k-1). \end{aligned}$$

In fact, all of the above formulas can be recovered in a similar way from Theorems 7.1.1, 7.1.2, and 7.1.3. In order to obtain a general formula for $w(2^k \pm j)$, similar formulas can be computed using this same technique:

$$\begin{aligned} w(2^k \pm 3) &= 4(k-4) + 5, \quad k \geq 4 \\ w(2^k \pm 5) &= 4(k-4) + 3, \quad k \geq 4 \\ w(2^k \pm 7) &= 8(k-5) + 9, \quad k \geq 5 \\ w(2^k \pm 9) &= 8(k-5) + 7, \quad k \geq 5 \\ w(2^k \pm 11) &= 6(k-6) + 8, \quad k \geq 6 \\ w(2^k \pm 13) &= 10(k-6) + 12, \quad k \geq 6 \\ w(2^k \pm 15) &= 12(k-6) + 13, \quad k \geq 6 \\ w(2^k \pm 17) &= 12(k-6) + 11, \quad k \geq 6, \end{aligned}$$

$$\begin{aligned} w(2^k \pm 19) &= 10(k-6) + 8, \quad k \geq 6 \\ w(2^k \pm 21) &= 6(k-6) + 4, \quad k \geq 6 \\ w(2^k \pm 23) &= 14(k-7) + 18, \quad k \geq 7 \\ w(2^k \pm 25) &= 18(k-7) + 22, \quad k \geq 7 \\ w(2^k \pm 27) &= 12(k-7) + 16, \quad k \geq 7 \\ w(2^k \pm 29) &= 18(k-7) + 20, \quad k \geq 7 \\ w(2^k \pm 31) &= 16(k-7) + 17, \quad k \geq 7. \end{aligned}$$

The values of k for which these formulas are valid depend upon the row in the triangular array in which it starts. Since $w(n)$ is based upon $s(3n)$, we can then expect $w(2^k \pm (2j+1))$ will appear in the $\lfloor \log_2(3(2j+1)) \rfloor$-th row. Analysis of these examples leads to the following theorem.

Theorem 7.2.1. *If $j = 2^m \pm_1 1$ and $k \geq m+1$, then $w(2^k \pm_2 j) = 2w(j)(k-r) \mp_1 1$, where $r = m+1 = \lfloor \log_2(3j) \rfloor$.*

We use $\pm_1$ and $\pm_2$ to distinguish between the two possible sign options, and that the signs are independent of one another. This makes it clear that the sign in $2w(j)(k-r) \mp 1$ only depends on the sign of $j = 2^m \pm 1$, and not the sign in $w(2^k \pm j)$.

Proof. Let $j = 2^m + 1$. Then clearly $2^{m+1} < 3j < 2^{m+2}$, so that $r = m+1$. Then using Theorem 7.1.1

twice, first with $n = 2^{k-m} \pm 1$ and then again with $n = 1$, we find that

$$\begin{aligned} w(2^k \pm_2 (2^m + 1)) &= w(2^m(2^{k-m} \pm_2 1) \pm_2 1) \\ &= w(2^2(2^{k-m} \pm_2 1) \pm_2 1) + 2(m-2)w(2^{k-m} \pm_2 1) \\ &= w(2^{k-m+2} \pm 5) + 2(m-2)w(2^{k-m} \pm_2 1). \end{aligned}$$

Then by using Theorems 7.1.1, 7.1.2, and 7.1.3 with $n = 1$, and noting that $w(4 \pm 1) = 2$, we have

$$\begin{aligned} w(2^k \pm_2 (2^m + 1)) &= 4 + (4(k - m + 2 - 4) - 1) + 4(m-2) + 4(m-2)(k-m-2) \\ &= 4(m-1)(k-m-1) - 1 \\ &= 2 \cdot 2(m-1)(k-r) - 1. \end{aligned}$$

Similarly, if $j = 2^m - 1$, we have

$$\begin{aligned} w(2^k \pm_2 (2^m - 1)) &= w(2^m(2^{k-m} \pm_2 1) \mp_2 1) \\ &= w(2^2(2^{k-m} \pm_2 1) \mp_2 1) + 2(m-2)w(2^{k-m} \pm_2 1) \\ &= w(2^{k-m+2} \pm 3) + 2(m-2)w(2^{k-m} \pm_2 1). \end{aligned}$$

Then by using Theorems 7.1.1, 7.1.2, and 7.1.3 with $n = 1$, and noting that $w(4 \pm 1) = 2$, we have

$$\begin{aligned} w(2^k \pm_2 (2^m - 1)) &= 4 + (4(k - m + 2 - 4) + 1) + 4(m-2) + 4(m-2)(k-m-2) \\ &= 4(m-1)(k-m-1) + 1 \\ &= 2 \cdot 2(m-1)(k-r) + 1. \end{aligned}$$

Then observing that $w(j) = w(4 \pm 1) + 2(m-2) = 2(m-1)$ and writing in condensed notation, we obtain the desired result. □

These have all been reduced cases of $w(2^k n \pm j)$ with $n = 1$, but we want to generalize. Recall that the middle of a row occurs at 2^{r-1}. If we consider the third row, in which 3 and 5 appear, the middle value is 4. We rewrite some recurrences in a more suggestive way to make the pattern more obvious:

$$\begin{aligned} w(16n \pm 3) &= 2w(4n \pm 1) + w(n) = w(3)w(4n \pm 1) + (4-3)w(1)w(n), \\ w(16n \pm 5) &= 2w(4n \pm 1) - w(n) = w(5)w(4n \pm 1) + (4-5)w(1)w(n). \end{aligned}$$

This pattern generalizes into the following formula.

Theorem 7.2.2. *Let $r = r(j) = \lfloor \log_2(3j) \rfloor$. Then for all $n \in \mathbb{N}$ and $1 \leq j < 2^k/3$, we have*

$$w(2^k n \pm j) = w(j)w(2^{k-r+1}n \pm 1) + \operatorname{sgn}(2^{r-1} - j)w(|2^{r-1} - j|)w(n). \tag{7.4}$$

Proof. We will proceed by double induction, first on k and then on j. Fix n.

For the induction on k, we fix j, which means $2^r/3 < j < 2^{r+1}/3$ for some r, and k must be greater than r. For the base case, we set $k = r + 1$. Using (7.1) and (1.10), we have

$$\begin{aligned} w(2^{r+1}n \pm j) &= \frac{1}{2}s(2^{r+1} \cdot 3n \pm 3j) = \frac{1}{2}s(2^{r+1} - 3j)s(3n) + \frac{1}{2}s(3j)s(3n \pm 1) \\ &= s(2^{r+1} - 3j)w(n) + w(j)(w(4n \pm 1) - w(n)) \\ &= w(j)w(4n \pm 1) + w(n)(s(2^{r+1} - 3j) - w(j)). \end{aligned}$$

We need to show

$$s(2^{r+1} - 3j) - w(j) = \operatorname{sgn}(2^{r-1} - j)w(|2^{r-1} - j|).$$

There are three cases: either $2^r/3 < j < 2^{r-1}$, or $j = 2^{r-1}$, or $2^{r-1} < j < 2^{r+1}/3$. We consider the easiest case first. If $j = 2^{r-1}$, then

$$s(2^{r+1} - 3j) - w(j) = s(2^{r+1} - 3 \cdot 2^{r-1}) - w(2^{r-1}) = s(1) - 1 = 0 = \operatorname{sgn}(0)w(0).$$

Secondly, if $\frac{1}{3}2^r < j < 2^{r-1}$, we can rewrite j as $j = 2^{r-1} - m$, for some m with $1 \leq m \leq 2^{r-1} - \lfloor \frac{2^r}{3} \rfloor$. Using (7.1) we have

$$\begin{aligned} s(2^{r+1} - 3j) - w(j) &= s(2^{r+1} - 3(2^{r-1} - m)) - w(2^{r-1} - m) \\ &= s(2^{r-1} + 3m) - \frac{1}{2}s(2^{r-1}3 - 3m) \\ &= s(2^{r-1} - 3m) + s(3m) - \frac{1}{2}s(2^{r-1} - 3m)s(3) - \frac{1}{2}s(3m) \\ &= \frac{1}{2}s(3m) \\ &= w(m). \end{aligned}$$

Note $m = 2^{r-1} - j = |2^{r-1} - j|$, so that $s(2^{r+1} - 3j) - w(j) = \operatorname{sgn}(2^{r-1} - j)w(|2^{r-1} - j|)$.

For the last case, if $2^{r-1} < j < \frac{1}{3}2^{r+1}$, we rewrite j as $j = 2^{r-1} + m$, with $1 \leq m \leq \lfloor \frac{2^{r+1}}{3} \rfloor - 2^{r-1}$. Again, using (7.1) we have

$$\begin{aligned}
s(2^{r+1}-3j)-w(j) &= s(2^{r+1}-3(2^{r-1}+m))-w(2^{r-1}+m) \\
&= s(2^{r-1}-3m)-\frac{1}{2}s(2^{r-1}3+3m) \\
&= s(2^{r-1}-3m)-\frac{1}{2}s(2^{r-1}-3m)s(3)-\frac{1}{2}s(3m) \\
&= -\frac{1}{2}s(3m) = -w(m) \\
&= \operatorname{sgn}(2^{r-1}-j)w(|2^{r-1}-j|),
\end{aligned}$$

since $2^{r-1}-j<0$ and $2^{r-1}-j=-m$. This finishes the base case.

Now assume (7.4) is true for $k<K$. Then rearranging and using the induction hypothesis, we have

$$\begin{aligned}
w(2^K n \pm j) &= w(2^{K-1}(2n)\pm j) \\
&= w(j)w(2^{K-1+r+1}(2n)\pm 1)+\operatorname{sgn}(2^{r-1}-j)w(|2^{r-1}-j|)w(2n) \\
&= w(j)w(2^{K-r+1}n\pm 1)+\operatorname{sgn}(2^{r-1}-j)w(|2^{r-1}-j|)w(n).
\end{aligned}$$

Thus, (7.4) holds for all $k>r$ and fixed j.

We now induct on j. For the base case $j=1$, the statement is trivially true, as we have $w(2^k n\pm 1)=w(2^k n\pm 1)$. Assume (7.4) is true for all $j\le N$. We also assume $k>r(N)$ and that (7.4) holds for all $k>r(N)$. Then we need to show

$$w(2^{\ell+1}n\pm(N+1)) = w(N+1)w(4n\pm 1)+\operatorname{sgn}(2^{\ell-1}-(N+1))w(|2^{\ell-1}-(N+1)|)w(n),$$

where $\ell=r(N+1)$ denotes the row number of $N+1$. (Since the parameter j is not fixed, the row number is also not fixed, and we use $r(j)$ as a function of j.) Then we can rewrite $N+1$ as $N+1=2^{\ell-1}\pm m$ for some positive integer m with $1\le m<\frac{2^{\ell-1}}{3}$. We have

$$w(2^{\ell+1}n\pm(N+1)) = w(2^{\ell+1}n\pm(2^{\ell-1}\pm m)) = w(2^{\ell-1}(4n\pm 1)\pm m).$$

Now, we have $r(m) < \ell - 1$, so that (7.4) holds. Applying (7.4), we have

$$\begin{aligned} w(2^{\ell+1}n \pm (N+1)) &= w(m)w(2^{\ell-r(m)}(4n \pm 1) \pm 1) \\ &\quad + \operatorname{sgn}(2^{r(m)-1} - m)w(|2^{r(m)-1} - m|)w(4n \pm 1) \\ &= w(m)w(2^{\ell-r(m)+2}n \pm (2^{\ell-r(m)} \pm 1)) \\ &\quad + \operatorname{sgn}(2^{r(m)-1} - m)w(|2^{r(m)-1} - m|)w(4n \pm 1). \end{aligned}$$

We again apply (7.4), but this time on $w(2^{\ell-r(m)+2}n \pm (2^{\ell-r(m)} \pm 1))$. If we let $a = 2^{\ell-r(m)} \pm 1$, note that $r(a) = \ell - r(m) + 1$. This means $2^{r(a)-1} - a = \mp 1$. So we have

$$\begin{aligned} w(2^{\ell+1}n \pm (N+1)) &= w(m)(w(4n \pm 1)w(2^{\ell-r(m)} \pm 1) + \operatorname{sgn}(\mp 1)w(|\mp 1|)w(n)) \\ &\quad + \operatorname{sgn}(2^{r(m)-1} - m)w(|2^{r(m)-1} - m|)w(4n \pm 1) \\ &= w(m)(w(4n \pm 1)w(2^{\ell-r(m)} \pm 1) + \operatorname{sgn}(\mp 1)w(n)) \\ &\quad + \operatorname{sgn}(2^{r(m)-1} - m)w(|2^{r(m)-1} - m|)w(4n \pm 1) \\ &= (w(m)w(2^{\ell-r(m)} \pm 1) + \operatorname{sgn}(2^{r(m)-1} - m)w(|2^{r(m)-1} - m|))w(4n \pm 1) \\ &\quad + \operatorname{sgn}(\mp 1)w(m)w(n). \end{aligned}$$

Using (7.4), we have

$$w(m)w(2^{\ell-r(m)} \pm 1) + \operatorname{sgn}(2^{r(m)-1} - m)w(|2^{r(m)-1} - m|) = w(2^{\ell-1} \pm m),$$

and noting that $2^{\ell-1} \pm m = N + 1$, we have

$$\begin{aligned} w(2^{\ell+1}n \pm (N+1)) &= w(2^{\ell-1} \pm m)w(4n \pm 1) + \operatorname{sgn}(\mp 1)w(m)w(n) \\ &= w(N+1)w(4n \pm 1) + \operatorname{sgn}(\mp 1)w(m))w(n). \end{aligned}$$

Note that $|m| = |2^{\ell-1} - (N+1)|$. If $N+1 = 2^{\ell-1} + m$ then $2^{\ell-1} - (N+1))$ is negative, and if $N+1 = 2^{\ell-1} - m$, then $2^{\ell-1} - (N+1))$ is positive, so that

$$w(2^{\ell+1}n \pm (N+1)) = w(N+1)w(4n \pm 1) + \operatorname{sgn}(2^{\ell-1} - (N+1))w(|2^{\ell-1} - (N+1)|)w(n).$$

Thus, if (7.4) holds for N, it also holds for $N + 1$. □

The terms which involve $2^{r(j)-1} - j$ seem very complicated, but in some way, they measure the distance to the middle of the row and the $\text{sgn}(2^{r(j)-1} - j)$ keeps track of whether it is to the left or right of the middle of the row.

Remark 8. Since the case $j = 1$ gives no new information, Theorem 7.1.1 is necessary and stands by itself. Also note Theorems 7.1.2 and 7.1.3 are special cases. The previous computations of $w(2^k \pm j)$ could also be found by setting $n = 1$ and the value for j, and using $w(2^{k-r+1} \pm 1) = 2(k - r)$. With a bit of rewriting, Theorem 7.2.1 is also a special case of Theorem 7.2.2. In (7.4), let $n = 1$ and $j = 2^m \pm 1$. Then $r(j) = m+1$, and using Theorem 7.1.1 we have

$$\begin{aligned}
w(2^k \pm j) &= w(j)w(2^{k-r+1} \pm 1) + \text{sgn}(2^{r-1} - j)w(|2^{r-1} - j|) \\
&= w(j)w(2^{k-m} \pm 1) + \text{sgn}(2^m - (2^m \pm 1))w(|2^m - (2^m \pm 1)|) \\
&= w(j)(w(4 \pm 1) + 2(k - m - 2)) \mp w(1) \\
&= w(j)(2 + 2(k - m - 2)) \mp 1 \\
&= w(j)(2 + 2(k - (m + 1) - 1)) \mp 1 \\
&= 2w(j)(k - r) \mp 1.
\end{aligned}$$

Chapter 8

GCD's of Consecutive Terms

The Stern sequence has the property that $\gcd(s(n), s(n+1)) = 1$ for all n. In addition to this property, we have for any a and b relatively prime, there is an n such that $s(n) = a$ and $s(n+1) = b$. We ask, does $w(n)$ inherit any of these nice properties as well?

8.1 Two Consecutive Terms

Upon examination of Table 5.1, we see it is not true that $\gcd(w(n), w(n+1)) = 1$ for all n. For example, $w(5) = w(6) = 2$. If consecutive terms are not relatively prime, what values may the greatest common divisor take? Are all values possible?

Theorem 8.1.1. *For any integer $a > 0$, there is an n such that* $\gcd(w(n), w(n+1)) = a$.

Proof. Let $n = (4^a - 1)/3$. Then using the Stern sequence to evaluate this, and recalling $s(2^r + 1) = r + 1$ and $s(2^r - 1) = r$, we have

$$w\left(\frac{4^a - 1}{3}\right) = \frac{1}{2}s(2^{2a} - 1) = \frac{1}{2}(2a) = a,$$

and

$$\begin{aligned} w\left(\frac{4^a - 1}{3} + 1\right) &= \frac{1}{2}s(2^{2a} - 1 + 3) = \frac{1}{2}s(2^{2a} + 2) \\ &= \frac{1}{2}s(2^{2a-1} + 1) = \frac{1}{2}(2a - 1 + 1) = a. \end{aligned}$$

Thus, $\gcd(w(n), w(n+1)) = a$. □

Remark 9. Let $g(n) := \gcd(w(n), w(n+1))$. While such an n exists, it need not be unique. For example, $w(5) = w(6) = 2$, and $w(7) = 4$, so that for $n = 5$ and $n = 6$, we have $g(5) = g(6) = 2$.

Considering consecutive terms, we can also ask, what is the probability $g(n) = 1$? Using Mathematica to compute these values, we have Table 8.1, which gives the empirical probability that $w(n)$ and $w(n+1)$ are relatively prime for $n \in [1, 2^N]$ and varying N. Besides the second row where the probability is 1, the next

Table 8.1: Percentage that $g(n) = 1$ for $n \in [1, 2^N]$

N	$P(g(n)) = 1$	N	$P(g(n) = 1)$	N	$P(g(n) = 1)$
2	1.0	8	0.7851	14	0.7904
3	0.75	9	0.7715	15	0.7913
4	0.75	10	0.7900	16	0.7939
5	0.75	11	0.8012	17	0.7949
6	0.8125	12	0.7988	18	0.7938
7	0.8125	13	0.7931	19	0.7929

highest probability seems to be 0.8125, with minimum values of 0.75. As N grows larger, the probability seems to oscillate quite a bit, but stays chiefly between 0.79 and 0.81. Does the probability converge to any value as N increases, or is it periodic? The data in Table 8.1 suggests the value stays around 0.793. We can also look at this data in arithmetic progressions modulo 4. First we note a special case modulo 4.

Theorem 8.1.2. *For all natural numbers n, we have* $\gcd(w(4n), w(4n+1)) = \gcd(w(4n+3), w(4n+4)) = 1$.

Proof. This follows from using (1.8) and (1.9), as well as the fact that consecutive terms in the Stern sequence are relatively prime. We have

$$\begin{aligned} \gcd(w(4n), w(4n+1)) &= \gcd\left(\frac{1}{2}s(3n), \frac{1}{2}s(3n) + s(3n+1)\right) \\ &= \gcd\left(\frac{1}{2}s(3n), s(3n+1)\right) = 1, \\ \gcd(w(4n+3), w(4n+4)) &= \gcd\left(\frac{1}{2}s(3n+3), \frac{1}{2}s(3n+3) + s(3n+2)\right) \\ &= \gcd\left(\frac{1}{2}s(3n+3), s(3n+2)\right) = 1. \end{aligned}$$

□

Looking at the other possible pairs in the progression modulo 4, we have Table 8.2. As N increases, these

Table 8.2: Percentage that $g(4n+1) = 1$, $g(4n+2) = 1$ for $n \in [1, 2^N]$

N	$P(g(4n+1) = 1)$	$P(g(4n+2) = 1)$	N	$P(g(4n+1) = 1)$	$P(g(4n+2) = 1)$
4	0.5625	0.6250	10	0.5986	0.5957
5	0.5625	0.6563	11	0.5806	0.5913
6	0.5313	0.5781	12	0.5823	0.5786
7	0.5391	0.5391	13	0.5814	0.5836
8	0.5625	0.5938	18	0.5859	0.5858
9	0.6016	0.5996	20	0.5868	0.5865

values seem to oscillate around 0.58. Note that since $g(4n) = g(4n+3) = 1$, and $g(4n+1)$ and $g(4n+2)$ are around 0.58, then $g(n)$ is around $(1 + 0.58)/2 = 0.79$, as expected.

8.2 Three and Four Consecutive Terms

What about the greatest common divisor of three consecutive terms? Generalizing $g(n)$, let

$$g_k(n) := \gcd(w(n), w(n+1), \ldots, w(n+k)).$$

Using Theorem 8.1.2, we have the following corollaries.

Corollary 8.2.1. *For all natural numbers n,*

$$g_2(4n) = g_2(4n+2) = g_2(4n+3).$$

Proof. Since $\gcd(w(4n), w(4n+1)) = \gcd(w(4n+3), w(4n+4)) = 1$, then the greatest common divisor when adding a consecutive third term is still 1. □

Note that $g_2(4n+1)$ does not appear, since we have that $\gcd(w(5), w(6), w(7)) = 2$.

Corollary 8.2.2. *For all natural numbers n, we have $g_3(n) = 1$.*

Proof. If three terms are relatively prime, then when adding a fourth term, the greatest common factor is still 1. Since

$$\begin{aligned}\gcd(w(4n), w(4n+1), w(4n+2)) &= \gcd(w(4n+2), w(4n+3), w(4n+4)) \\ &= \gcd(w(4n+3), w(4n+4), w(4n+5)) = 1,\end{aligned}$$

then any 4 consecutive terms in an arithmetic progression modulo 4 are relatively prime, so that $g_3(n) = 1$ holds for all n. □

Then by the process of elimination, the greatest common divisor of $w(4n+1)$, $w(4n+2)$, and $w(4n+3)$) is not always 1. However, the frequency with which for this 3-tuple is not relatively prime is quite small, as evidenced by Table 8.3. Most of the time it appears that three consecutive terms are relatively prime. When they are not relatively prime, data suggests the greatest common divisor is 2. Since the empirical probability that these three terms are relatively prime is relatively high, is there a branch of this arithmetic progression where the common divisor is 2 more often? Is $g_2(4n+1)$ unbounded? If we examine $g_2(8n+1)$ and $g_2(8n+5)$, the empirical probability that the greatest common divisor is 1 does not change from that of $g_2(4n+1)$. However, considering $g_2(16n+1)$, $g_2(16n+5)$, $g_2(16n+9)$, and $g_2(16n+13)$, we notice $g_2(16n+1)$ and $g_2(16n+13)$ seem to only take the value 1. This leads us to the following theorem.

Table 8.3: Percentage that $g_2(4n+1)=1$ for $n \in [1, 2^N]$

N	$P(g_2(4n+1)=1)$	N	$P(g_2(4n+1)=1)$
5	0.84375	13	0.833374
6	0.796875	14	0.836121
7	0.804688	15	0.834686
8	0.839844	16	0.832626
9	0.849609	17	0.832298
10	0.838867	18	0.833172
11	0.827637	19	0.833769
12	0.827881	20	0.833632

Theorem 8.2.3. *For all $n \in \mathbb{N}$, we have*

$$\gcd(w(16n+1), w(16n+2), w(16n+3)) = \gcd(w(16n+13), w(16n+14), w(16n+15)) = 1.$$

Proof. First consider $\gcd(w(16n+1), w(16n+2), w(16n+3))$. Using (7.2), (7.3), (1.6) and (1.8), we have

$$\begin{aligned}
\gcd(w(16n+1), w(16n+2), w(16n+3)) &= \gcd(w(4n+1)+4w(n), w(8n+1), 2w(4n+1)+w(n)) \\
&= \gcd(w(4n+1)+4w(n), w(4n+1)+2w(n), 2w(4n+1)+w(n)) \\
&= \gcd(2w(n), w(4n+1)+2w(n), 2w(4n+1)+w(n)) \\
&= \gcd(2w(n), w(4n+1), 2w(4n+1)+w(n)) \\
&= \gcd(w(4n+1), w(n)) \\
&= \gcd(w(n)+s(3n+1), w(n)) \\
&= \gcd(s(3n+1), w(n)) \\
&= \gcd(s(3n+1), \frac{1}{2}s(3n)) \\
&= 1.
\end{aligned}$$

Now consider $\gcd(w(16n+13), w(16n+14), w(16n+15))$. Similarly, using (7.2), (7.3), (1.6) and (1.9), we have

$$\begin{aligned}
\gcd(w(16n+13), w(16n+14), w(16n+15)) &= \gcd(2w(4n+3)+w(n+1), w(8n+7), w(4n+3)+4w(n+1)) \\
&= \gcd(2w(4n+3)+w(n+1), w(4n+3)+2w(n+1), w(4n+3)+4w(n+1)) \\
&= \gcd(2w(4n+3)+w(n+1), w(4n+3)+2w(n+1), 2w(n+1)) \\
&= \gcd(2w(4n+3)+w(n+1), w(4n+3), 2w(n+1)) \\
&= \gcd(w(n+1), w(4n+3), 2w(n+1)) \\
&= \gcd(w(n+1), w(4n+3)) \\
&= \gcd(w(n+1), w(n+1)+s(3n+2)) \\
&= \gcd(w(n+1), s(3n+2)) \\
&= \gcd\left(\frac{1}{2}s(3n+3), s(3n+2)\right) \\
&= 1. \qquad \square
\end{aligned}$$

If we examine more branches of $\gcd(4n+1, 4n+2, 4n+2)$, with higher powers of 2 within the arithmetic progression, the data suggests $g_2(64n+9)$, $g_2(64n+53)$, $g_2(256n+85)$, and $g_2(256n+213)$ are always 1.

Chapter 9

$w(n)$ Modulo 2

The Stern sequence has a combinatorial interpretation in terms of a generalized binary representation. The corresponding generating functions for these generalized binary representations will also be periodic (see [1]). Finding an explicit form of the generating function for $w(n)$ is not easy, and there is no obvious combinatorial interpretation independent of $s(n)$. As a means of finding a combinatorial interpretation in terms of generalized binary representations, we examine to see if $w(n)$ is periodic modulo 2.

9.1 Two Consecutive Terms Modulo 2

In this chapter, we consider the frequency of the pairs $(w(n) \pmod 2, w(n+1) \pmod 2)$, and we show that each of $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ has a limiting frequency.

Theorem 9.1.1. *The probability of* $(w(n), w(n+1)) = (1,0) \pmod 2$, *and* $(w(n), w(n+1)) = (0,1) \pmod 2$ *is* $1/3$, *while the probability of* $(w(n), w(n+1)) = (0,0) \pmod 2$ *and* $(w(n), w(n+1)) = (1,1) \pmod 2$ *is* $1/6$.

Surprisingly, the proof of this depends not only on the parity of $w(n)$ and $w(n+1)$, but also on the parity of $w(2n+1)$. We first need several helpful lemmas and theorems. The methodology used to prove this theorem is taken from [27]. We first show only six cases are possible and then use matrix theory to show that each case is equally likely.

Lemma 9.1.2. *For all* n, *it is never the case that* $w(n) \equiv w(n+1) \equiv w(2n+1) \pmod 2$.

Proof. First note that $w(n) \equiv w(n+1) \equiv w(2n+1) \pmod 2$ is equivalent to $s(3n) \equiv s(3n+3) \equiv s(6n+3) \pmod 4$. If n is even, we write $n = 2m$, and we have

$$s(3n) = s(6m) = s(3m),$$

$$s(3n+3) = s(6m+3) = s(3m+1) + s(3m+2),$$

$$s(6n+3) = s(12m+3) = s(3m) + 2s(3m+1).$$

If $s(3n) \equiv 0 \pmod 4$, then $s(6n+3)$ will be congruent to 2 modulo 4 since $s(3m+1)$ is odd. Similarly, if $s(3n) \equiv 2 \pmod 4$, then $s(6n+3) \equiv 2+2 \equiv 0 \pmod 4$.

If n is odd, we write $n = 2m+1$, and we have

$$\begin{aligned} s(3n) &= s(6m+3) = s(3m+1) + s(3m+2), \\ s(3n+3) &= s(6m+6) = s(3m+3), \\ s(6n+3) &= s(12m+9) = s(3m+3) + 2s(3m+2). \end{aligned}$$

If $s(3n+3) \equiv 0 \pmod 4$, then $s(6n+3)$ will be congruent to 2 modulo 4 since $s(3m+1)$ is odd. Similarly, if $s(3n+3) \equiv 2 \pmod 4$, then $s(6n+3) \equiv 2+2 \equiv 0 \pmod 4$.

Thus, if $w(n)$ and $w(n+1)$ are even, then $w(2n+1)$ will be odd, and if $w(n)$ and $w(n+1)$ are odd, then $w(2n+1)$ will be even. □

Next, we show the remaining 6 possible outcomes for parity are equally likely.

Definition 9.1.3. Let $A_k(\ell, r) := \#\{\ell \cdot 2^r \le n < (\ell+1) \cdot 2^r : (w(n), w(n+1), w(2n+1)) \equiv a_k \pmod 2\}$, where

$$\begin{aligned} a_1 &= (0,0,1), \quad a_2 = (0,1,0), \quad a_3 = (0,1,1), \\ a_4 &= (1,0,0), \quad a_5 = (1,0,1), \quad a_6 = (1,1,0). \end{aligned}$$

We note that $A_k(\ell, r)$ counts the number of times each 3-tuple occurs, between scaled powers of 2. These counting functions satisfy recurrence relationships, and we have the following theorem.

Theorem 9.1.4. *For all $r \ge 0$ and $\ell \ge 0$,*

$$\begin{aligned} A_1(\ell, r+1) &= A_2(\ell, r) + A_4(\ell, r), \\ A_2(\ell, r+1) &= A_2(\ell, r) + A_6(\ell, r), \\ A_3(\ell, r+1) &= A_1(\ell, r) + A_3(\ell, r), \\ A_4(\ell, r+1) &= A_4(\ell, r) + A_6(\ell, r), \\ A_5(\ell, r+1) &= A_1(\ell, r) + A_5(\ell, r), \\ A_6(\ell, r+1) &= A_3(\ell, r) + A_5(\ell, r), \end{aligned}$$

with $A_1(1,0) = A_2(1,0) = A_3(1,0) = A_4(1,0) = A_5(1,0) = 0$, and $A_6(1,0) = 1$. We also have $A_k(\ell, 0) = 1$

for some k and $A_j(\ell, 0) = 0$ for all $j \neq k$.

Proof. We proceed by induction. For $A_k(1,0)$, we only have $(w(1), w(2), w(3)) \equiv (1,1,0) \pmod 2$, so that $A_6(1,0) = 1$ and $A_k(1,0) = 0$ for $k \neq 6$. For $A_k(1,1)$, we have $2 \leq n < 4$, and

$$(w(2), w(3), w(5)) \equiv (1,0,0) \pmod 2 \quad \text{and} \quad (w(3), w(4), w(6)) \equiv (0,1,0) \pmod 2,$$

so that the above recurrences hold for $A_k(1,1)$. Also note

$$A_k(\ell, 0) = \#\{\ell \leq n < \ell + 1 : (w(n), w(n+1), w(2n+1)) \equiv a_k \pmod 2\},$$

which means $A_k(\ell, 0) = 1$ for some k and $A_j(\ell, 0) = 0$ for all $j \neq k$.

Now assume the recurrences are true up to some r. First note

$$w(4n+1) = w(n) + s(3n+1) \equiv w(n) + 1 \pmod 2,$$

so that $w(4n+1)$ will have opposite parity to $w(n)$, since $s(3n+1)$ is always odd. Similarly, we have

$$w(4n+3) = w(n+1) + s(3n+2) \equiv w(n+1) + 1 \pmod 2.$$

Then if $(w(n), w(n+1), w(2n+1)) \equiv (A, B, C)$ modulo 2, we have

$$(w(2n), w(2n+1), w(4n+1)) \equiv (A, C, A+1) \pmod 2, \quad \text{and} \tag{9.1}$$

$$(w(2n+1), w(2n+2), w(4n+3)) \equiv (C, B, B+1) \pmod 2. \tag{9.2}$$

This means if we know $(w(n), w(n+1), w(2n+1)) \equiv a_k$, then we can determine what happens at the next level up, for higher values. For example, if $(w(n), w(n+1), w(2n+1)) \equiv (0,0,1)$, then

$$(w(2n), w(2n+1), w(4n+1)) \equiv (0,1,1), \quad \text{and}$$

$$(w(2n+1), w(2n+2), w(4n+3)) \equiv (1,0,1).$$

We can also use this to go the other direction and calculate $A_k(\ell, r+1)$, based on which $A_j(\ell, r)$ lead into it. We have $A_k(\ell, r+1)$ counts the $\ell \cdot 2^{r+1} \leq n < (\ell+1) \cdot 2^{r+1}$ such that $(w(n), w(n+1), w(2n+1)) \equiv a_k$, and if $n = 2m$ or $n = 2m+1$, this reduces to counting the $\ell \cdot 2^r \leq m < (\ell+1) \cdot 2^r$ such that $(w(2m), w(2m+1), w(4m+1)) \equiv a_k$ or $(w(2m+1), w(2m+2), w(4m+3)) \equiv a_k$. Then using (9.1), we find the $A_j(\ell, r)$ which satisfy these equivalences, and this gives us the recurrences. The recursions can be drawn into a planar graph, as shown in Figure 9.1.

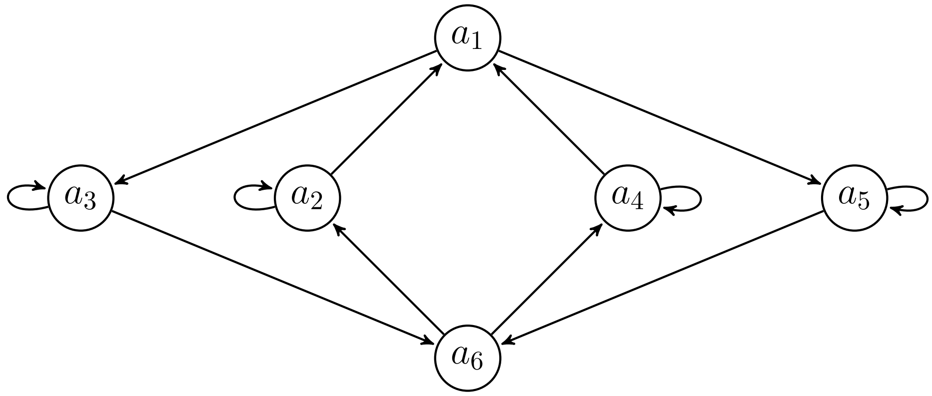


Figure 9.1: Graph of Recursions

In Figure 9.1, an arrow from a state a_i to another state a_j occurs when n is replaced by $2n$ or $2n+1$ and the state a_j is the outcome. We recover the recurrences by taking a state a_k and finding the a_i's which have arrows leading to a_k.

For example, to find $A_1(\ell, r+1)$, we consider a_1 in Figure 9.1 and see that a_2 and a_4 have arrows leading to a_1. This means $A_1(\ell, r+1) = A_2(\ell, r) + A_4(\ell, r)$. We do this with each of the $A_k(\ell, r+1)$, and in this way recover all of the recurrences. Thus, if the recurrences are true for integers less than r, this implies they are also true for $r+1$, and thus they are true for all $r \geq 0$. □

We can define a transition matrix based off these recurrences. We have

$$\begin{pmatrix} A_1(\ell, r+1) \\ A_2(\ell, r+1) \\ A_3(\ell, r+1) \\ A_4(\ell, r+1) \\ A_5(\ell, r+1) \\ A_6(\ell, r+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_1(\ell, r) \\ A_2(\ell, r) \\ A_3(\ell, r) \\ A_4(\ell, r) \\ A_5(\ell, r) \\ A_6(\ell, r) \end{pmatrix}.$$

Iterating the recurrence relations, we also have

$$\begin{pmatrix} A_1(\ell,r) \\ A_2(\ell,r) \\ A_3(\ell,r) \\ A_4(\ell,r) \\ A_5(\ell,r) \\ A_6(\ell,r) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}^r \begin{pmatrix} A_1(\ell,0) \\ A_2(\ell,0) \\ A_3(\ell,0) \\ A_4(\ell,0) \\ A_5(\ell,0) \\ A_6(\ell,0) \end{pmatrix}.$$

We can also interpret this information in terms of random walks. Let X be a random variable. If we start at $X = a_k$, we consider the probability of being at another state $X = a_j$ after r steps. For $r = 0$, we have $Pr(X = a_6) = 1$ while $Pr(X = a_j) = 0$ for $j \neq 6$. For $r = 1$, we have $Pr(X = a_2) = \frac{1}{2}$ and $Pr(X = a_4) = \frac{1}{2}$, while the others have probability 0. After r steps, there are 2^r total outcomes, so that the probability of being in state a_k is $A_k(\ell,r)/(2^r)$, which means the probability of a_k after $(r+1)$ steps is

$$\frac{A_k(\ell,r+1)}{2^{r+1}} = \frac{1}{2}\left(\frac{A_i(\ell,r)+A_j(\ell,r)}{2^r}\right)$$

for suitable i and j. Let

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

be the transition matrix. This matrix has characteristic polynomial

$$p(x) = 16x^6 - 32x^5 + 24x^4 - 8x^3 - 3x^2 + 4x - 1 = 16(x-1)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^2\left(x-\frac{1+i\sqrt{7}}{4}\right)\left(x-\frac{1-i\sqrt{7}}{4}\right),$$

and eigenvalues of $1/2$, $-1/2$, 1, and $\frac{1\pm i\sqrt{7}}{4}$. As a side note, the eigenvalue with the largest modulus is 1. The next largest modulus is $|\frac{1\pm i\sqrt{7}}{4}| = \frac{\sqrt{2}}{2}$. These eigenvalues are important for a bound on the matrix entries. This means

$$\begin{pmatrix} Pr(X = a_1(r+1)) \\ Pr(X = a_2(r+1)) \\ Pr(X = a_3(r+1)) \\ Pr(X = a_4(r+1)) \\ Pr(X = a_5(r+1)) \\ Pr(X = a_6(r+1)) \end{pmatrix} = M \cdot \begin{pmatrix} Pr(X = a_1(r)) \\ Pr(X = a_2(r)) \\ Pr(X = a_3(r)) \\ Pr(X = a_4(r)) \\ Pr(X = a_5(r)) \\ Pr(X = a_6(r)) \end{pmatrix}.$$

We use some standard matrices results, combined into the following lemma, which can be found in [21, p. 516] and [31, p. 7].

Lemma 9.1.5. *Let A be an $n \times n$ non-negative matrix, with $A^m > 0$ for some $m \geq 1$. Suppose the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_t$, with $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_t|$. If $|\lambda_2| = |\lambda_3|$, then we stipulate the multiplicity m_2 of λ_2 is at least as great as that of λ_3 and of any other eigenvalue having the same modulus as λ_2. Then we have*

$$\lim_{r\to\infty} A^r = \lambda_1^r x y^T + O(r^s |\lambda_2|^r),$$

where x and y are positive right and left eigenvectors corresponding to λ_1 with $x^T y = 1$, and $s = m_2 - 1$.

We then have the following theorem.

Theorem 9.1.6. *For $k = 1, 2, \ldots 6$, and for sufficiently large r, we have*

$$\frac{A_k(\ell, r)}{2^r} = \frac{1}{6} + O\left(\rho^r\right),$$

where $\rho = |(1 + i\sqrt{7})/4| = \sqrt{2}/2$.

Proof. First note the transition matrix M is non-negative. Taking powers of M, we have

$$M^4 = \begin{pmatrix} \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} & \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} \end{pmatrix},$$

so that each entry is positive, meaning $M^4 > 0$. We have $\lambda_1 = 1$, $|\lambda_2| = |(1 + i\sqrt{7})/4| = \sqrt{2}/2$ and $s = 0$. The eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$ are $x = (1, 1, 1, 1, 1, 1)^T$ and $y = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})^T$.

We have xy^T is a uniform matrix with $1/6$ for each entry. By Lemma 9.1.5, the probability matrix M converges to a uniform matrix with $1/6$ for each entry. This means

$$\frac{A_k(\ell, r)}{2^r} = \frac{1}{6} + O(\rho^r),$$

where $\rho = |(1 + i\sqrt{7})/4| = \sqrt{2}/2$. □

This does not prove each possibility is equally likely if we go up to an arbitrary N. We now define

$$\Delta_k(N) := \#\{n < N : (w(n), w(n+1), w(2n+1)) \equiv a_k \pmod 2\}.$$

We want to show

$$\lim_{N\to\infty} \frac{\Delta_k(N)}{N} = \frac{1}{6},$$

which would mean

$$(w(n), w(n+1), w(2n+1)) \equiv a_k \pmod 2 \quad \text{for } k = 1, 2, \ldots, 6,$$

are uniformly distributed across the 6 possibilities.

Theorem 9.1.7. *The 6 possible parity outcomes for* $(w(n), w(n+1), w(2n+1)) \mod 2$ *are equally likely with probability* $1/6$.

Proof. Fix k, and write N in terms of its binary expansion: $N = 2^{r_1} + 2^{r_2} + 2^{r_3} + \cdots + 2^{r_\nu}$. We then take the interval $I = [0, N)$ and break it up into the following subintervals:

$$I = [0, 2^{r_1}) \cup [2^{r_1}, 2^{r_1} + 2^{r_2}) \cup \ldots \cup [2^{r_1} + 2^{r_2} + \cdots + 2^{r_{\nu-1}}, 2^{r_1} + 2^{r_2} + 2^{r_3} + \cdots + 2^{r_\nu}).$$

We rewrite each subinterval as

$$I_j := [2^{r_j}(2^{r_1 - r_j} + 2^{r_2 - r_j} + \cdots + 2^{r_{j-1} - r_j}), 2^{r_j}(2^{r_1 - r_j} + 2^{r_2 - r_j} + \cdots + 2^{r_{j-1} - r_j} + 1),$$

for $j \geq 2$, with $I_1 = [0 \cdot 2^{r_1}, 1 \cdot 2^{r_1})$, so that we can use Theorem 9.1.6 on each I_j. For simplification, we define $\ell_j := 2^{r_1 - r_j} + 2^{r_2 - r_j} + \cdots + 2^{r_{j-1} - r_j}$ for $j \geq 2$, and $\ell_1 = 0$. We apply Theorem 9.1.6, and for $j \geq 1$ we have

$$\left| A_k(\ell_j, r_j) - \frac{2^{r_j}}{6} \right| < c_j 2^{r_j} \rho^{r_j} = c_j (\sqrt{2})^{r_j}.$$

Let $c = \max c_j$. We then have

$$\begin{aligned}
\left|\Delta_k(N) - \frac{N}{6}\right| &= \left|\sum_{j=1}^{\nu} A_k(\ell_j, r_j) - \left(\frac{2^{r_1}}{6} + \frac{2^{r_2}}{6} + \cdots + \frac{2^{r_\nu}}{6}\right)\right| \\
&= \left|\sum_{j=1}^{\nu} A_k(\ell_j, r_j) - \frac{2^{r_j}}{6}\right| \\
&\leq \sum_{j=1}^{\nu} \left|A_k(\ell_j, r_j) - \frac{2^{r_j}}{6}\right| \\
&< \sum_{j=1}^{\nu} c_j(\sqrt{2})^{r_j} \\
&< c\sum_{j=1}^{\nu}(\sqrt{2})^{r_j} \\
&< c \cdot (\sqrt{2}^{r_1} + \sqrt{2}^{r_1 - 1} + \cdots + 1) \\
&< c\frac{(\sqrt{2})^{r_1+1} - 1}{\sqrt{2} - 1} \\
&< c'(\sqrt{2})^{r_1}.
\end{aligned}$$

Since N is roughly the size of 2^{r_1}, we have $r_1 \approx \log N / \log 2$, so that $(\sqrt{2})^{r_1} = c'' N^{\frac{1}{2}}$. This implies

$$\left|\Delta_k(N) - \frac{N}{6}\right| < CN^{\frac{1}{2}},$$

for some C. Thus for any N, we have

$$\frac{\Delta_k(N)}{N} - \frac{1}{6} = O(N^{-\frac{1}{2}}). \qquad \square$$

Note we can use $\Delta_k(N)$ to count the number of $n < N$ such that $(w(n), w(n+1))$ is congruent to one of the six outcomes. We have

$$\begin{aligned}
\#\{n < N : (w(n), w(n+1)) \equiv (0,0) \pmod 2\} &= \Delta_1(N), \\
\#\{n < N : (w(n), w(n+1)) \equiv (0,1) \pmod 2\} &= \Delta_2(N) + \Delta_3(N), \\
\#\{n < N : (w(n), w(n+1)) \equiv (1,0) \pmod 2\} &= \Delta_4(N) + \Delta_5(N), \\
\#\{n < N : (w(n), w(n+1)) \equiv (1,1) \pmod 2\} &= \Delta_6(N).
\end{aligned}$$

Since $\Delta_k(N)/N$ is roughly $1/6$, we can use the above results to obtain the probabilities listed in Theorem

9.1.1. This also proves Theorem 9.1.1.

Recall that Theorem 9.1.1 states the probability of $(w(n), w(n+1)) = (1, 0) \mod 2$, and $(w(n), w(n+1)) = (0, 1) \pmod 2$ is 1/3, while the probability of $(w(n), w(n+1)) = (0, 0) \pmod 2$ and $(w(n), w(n+1)) = (1, 1) \pmod 2$ is 1/6.

We can also prove Theorem 9.1.1 via direct calculations.

Proof. We consider all consecutive pairs in arithmetic progressions of 8. We have

$$\begin{aligned} w(8n) &= w(n), \\ w(8n+1) &= w(4n+1) + 2w(n) = s(3n+1) + 3w(n), \\ w(8n+2) &= w(4n+1) = s(3n+1) + w(3n). \end{aligned}$$

We see that the parity of these three terms depend only on the parity of $w(n)$. We have $w(8n)$ and $w(8n+1)$ will have opposite parity, while $w(8n+1)$ and $w(8n+2)$ will have the same parity. We also have

$$\begin{aligned} w(8n+3) &= w(4n+1) + w(2n+1) - w(n) = w(2n+1) + s(3n+1), \\ w(8n+4) &= w(2n+1), \\ w(8n+5) &= w(4n+3) + w(2n+1) - w(n+1) = w(2n+1) + s(3n+2), \end{aligned}$$

so that these terms depend only on the parity of $w(2n+1)$. So $w(8n+3)$ and $w(8n+4)$ will have opposite parity, as will $w(8n+4)$ and $(8n+5)$. We then have

$$\begin{aligned} w(8n+6) &= w(4n+3) = w(n+1) + s(3n+2), \\ w(8n+7) &= w(4n+3) + 2w(n+1) = s(3n+2) + 3w(n+1), \\ w(8n+8) &= w(n+1). \end{aligned}$$

These terms only depend on the parity of $w(n+1)$, and $w(8n+6)$ and $w(8n+7)$ will have the same party, while $w(8n+7)$ and $w(8n+8)$ will have opposite parity.

By Lemma 9.1.2, there are only 6 possible cases for the parity of $w(n)$, $w(n+1)$, and $w(2n+1)$, and by Theorem 9.1.7 these six outcomes are all equally likely. Going through all six cases of possible parities, we can then write out all of the possible strings modulo 2, count them, and compute their probability. These calculations are by brute force of case analysis, and we omit these details. In conclusion, there are 48 total possibilities. The ordered pairs $(0, 0)$ and $(1, 1)$ occur 8 times, so that their probability is 1/6, and $(0, 1)$ and

$(1, 0)$ occur 16 times, so that their probability is 1/3. $\square$

We can condense these pairs into strings of 9. The 6 possible values of $(w(8n), w(8n+1), \ldots, w(8n+8))$ are

$$011010110, \quad 011101001, \quad 011010001,$$

$$100101110, \quad 100010110, \quad 100101001.$$

We can see that there are at most 3 1's or 3 0's in a row.

9.2 Three or Four Consecutive Terms Modulo 2

Using the same information, we can also say something about the distribution of triple of consecutive terms modulo 2.

Corollary 9.2.1. *The set of* $\{(w(n), w(n+1), w(n+2))\}$ *is not uniformly distributed among the ordered triples modulo 2. Furthermore, the probability of* $(0, 0, 0)$ *and* $(1, 1, 1)$ *are each* $1/24$*,* $(0, 0, 1)$*,* $(1, 0, 0)$*,* $(0, 1, 1)$*, and* $(1, 1, 0)$ *are each* $1/8$*,* $(0, 1, 0)$ *and* $(1, 0, 1)$ *are each* $5/24$*.*

Proof. In the previous proof, we gave relationships for the parity of terms in arithmetic progressions of 8. We take this previous information in triples instead of pairs, and note that $w(8n+9) = 3w(n+1)+s(3n+4)$, so that the parity depends on $w(n+1)$ and $w(8n+8)$ and $w(8n+9)$ will have opposite parity. We can then do calculations of all 6 cases by brute force. We omit these details here.

Again, there are 48 total possible outcomes. The probability of $(0, 0, 0)$ or $(1, 1, 1)$ is 1/24, the probability of $(0, 0, 1)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$ is 1/8, and the probability of $(0, 1, 0)$ and $(1, 0, 1)$ is 5/24. $\square$

Corollary 9.2.2. *Four consecutive terms are not uniformly distributed across the four-tuples modulo 2, with* $(0, 0, 0, 0)$ *and* $(1, 1, 1, 1)$ *never occurring.*

Proof. It is clear to see from the proof of Theorem 9.1.1 and the 6 possible strings, that $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$ will never occur. Consequently, four consecutive terms will never be uniformly distributed across all possible four-tuples modulo 2. $\square$

Chapter 10

Sums

Summing across a row in the diatomic array, Stern [34] observed and proved the sum of the values in the r-th row is $3^r + 1$. We rewrite this as the following lemma.

Lemma 10.0.3. *For $k \in \mathbb{N} \cup \{0\}$,*

$$\sum_{n=2^k}^{2^{k+1}-1} s(n) = 3^k. \tag{10.1}$$

D.H. Lehmer [24] also mentions this property in his paper, but only in summary of Stern's results from his 1858 paper.

Upon examining a list of values for $w(n)$, it was observed $w(n)$ followed a similar pattern in summing to powers of 3. This led to the following theorem.

Theorem 10.0.4. *For $k \in \mathbb{N} \cup \{0\}$,*

$$\sum_{n=2^k}^{2^{k+1}-1} w(n) = 3^k.$$

Proof. Let $T_k := \sum_{n=2^k}^{2^{k+1}-1} w(n)$. Then separating the summands over even and odd values, using the recurrence relation for $s(2n+1)$, then re-indexing, and separating into two sums, we have

$$2T_k = \sum_{n=2^k}^{2^{k+1}-1} s(3n) = \sum_{\substack{n=2^k \\ n \text{ even}}}^{2^{k+1}-1} s(3n) + \sum_{\substack{n=2^k \\ n \text{ odd}}}^{2^{k+1}-1} s(3n) = \sum_{\ell=2^{k-1}}^{2^k-1} s(3\ell) + \sum_{\ell=2^{k-1}}^{2^k-1} s(3(2\ell+1))$$

$$= \sum_{\ell=2^{k-1}}^{2^k-1} s(3\ell) + s(3\ell+1) + s(3\ell+2) = \sum_{\ell=3\cdot 2^{k-1}}^{3\cdot 2^k-1} s(\ell)$$

$$= \sum_{\ell=3\cdot 2^{k-1}}^{2^{k+1}} s(\ell) + \sum_{\ell=2^{k+1}+1}^{3\cdot 2^k-1} s(\ell) \tag{10.2}$$

$$= \sum_{\ell=2^k}^{3\cdot 2^{k-1}} s(\ell) + \sum_{\ell=3\cdot 2^k+1}^{2^{k+2}-1} s(\ell) \tag{10.3}$$

by using the symmetry of $s(n)$. Then adding (10.2) and (10.3) and rearranging the sums in a convenient way, we have

$$\begin{aligned} 4T_k = 2T_k + 2T_k &= \sum_{\ell=3\cdot 2^{k-1}}^{2^{k+1}} s(\ell) + \sum_{\ell=2^{k+1}+1}^{3\cdot 2^k-1} s(\ell) + \sum_{\ell=2^k}^{3\cdot 2^{k-1}} s(\ell) + \sum_{\ell=3\cdot 2^k+1}^{2^{k+2}-1} s(\ell) \\ &= \sum_{\ell=2^k}^{3\cdot 2^k-1} s(\ell) + s(3\cdot 2^{k-1}) + \sum_{\ell=3\cdot 2^k+1}^{2^{k+2}-1} s(\ell) \\ &= \sum_{\ell=2^k}^{2^{k+2}-1} s(\ell) = \sum_{\ell=2^k}^{2^{k+1}-1} s(\ell) + \sum_{\ell=2^{k+1}}^{2^{k+2}-1} s(\ell) \\ &= 3^k + 3^{k+1} = 4\cdot 3^k, \end{aligned}$$

which then implies the desired result. □

Remark 10. We define Σ^* to indicate that only half of the first and last values are taken, meaning

$$\sum_{n=a}^{b}{}^{*} f(n) = \sum_{n=a}^{b} f(n) - \frac{1}{2}f(a) - \frac{1}{2}f(b).$$

We then have

$$\sum_{n=2^k}^{2^{k+1}}{}^{*} s(n) = \sum_{n=2^k}^{2^{k+1}}{}^{*} w(n) = 3^k. \tag{10.4}$$

10.1 Order of Magnitude

Since the sums over intervals of powers of 2 are the same for both the Stern sequence and $w(n)$, we should expect the average values of the sequences to have the same magnitude. The Stern sequence has average order of $N^{\beta-1}$, where $\beta = \log_2 3$, and $w(n)$ indeed has the same order of magnitude for the average value. To see this, let $W(N) := \sum_{n=0}^{N} w(n)$. We have

$$W(2^{r+1}-1) = \sum_{n=0}^{2^{r+1}-1} w(n) = \sum_{j=0}^{r} \sum_{n=2^j}^{2^{j+1}-1} w(n) = \sum_{j=0}^{r} 3^j = \frac{3^{r+1}-1}{2}.$$

Noting that $W(2^{r+1}) = W(2^{r+1}-1) + w(2^{r+1}) = \frac{3^{r+1}-1}{2} + 1 = \frac{3^{r+1}+1}{2}$, we have for $2^r \le N < 2^{r+1}$,

$$\frac{3^r+1}{2} = W(2^r) \le W(N) \le W(2^{r+1}-1) = \frac{3^{r+1}-1}{2}.$$

Then since $(3^r+1)/2 > 3^{r+1}/6 > N^\beta/6$, and $(3^{r+1}-1)/2 < (N^\beta-1)/2 < N^\beta/2$, we have $W(N)/N \asymp N^{\beta-1}$.

10.2 Sums of $s(n) - w(n)$

The nice equality in (10.4) leads one to consider, what happens to the quantity

$$\sum_{n=2^k}^{2^k(1+t)} {}^{*} (s(n) - w(n)),$$

as t changes from 0 to 1. Letting k vary, and looking at the graphs given in Figure 10.1, we see that as k grows larger, the graphs seem to converge to the same shape. The graphs have been normalized. This leads

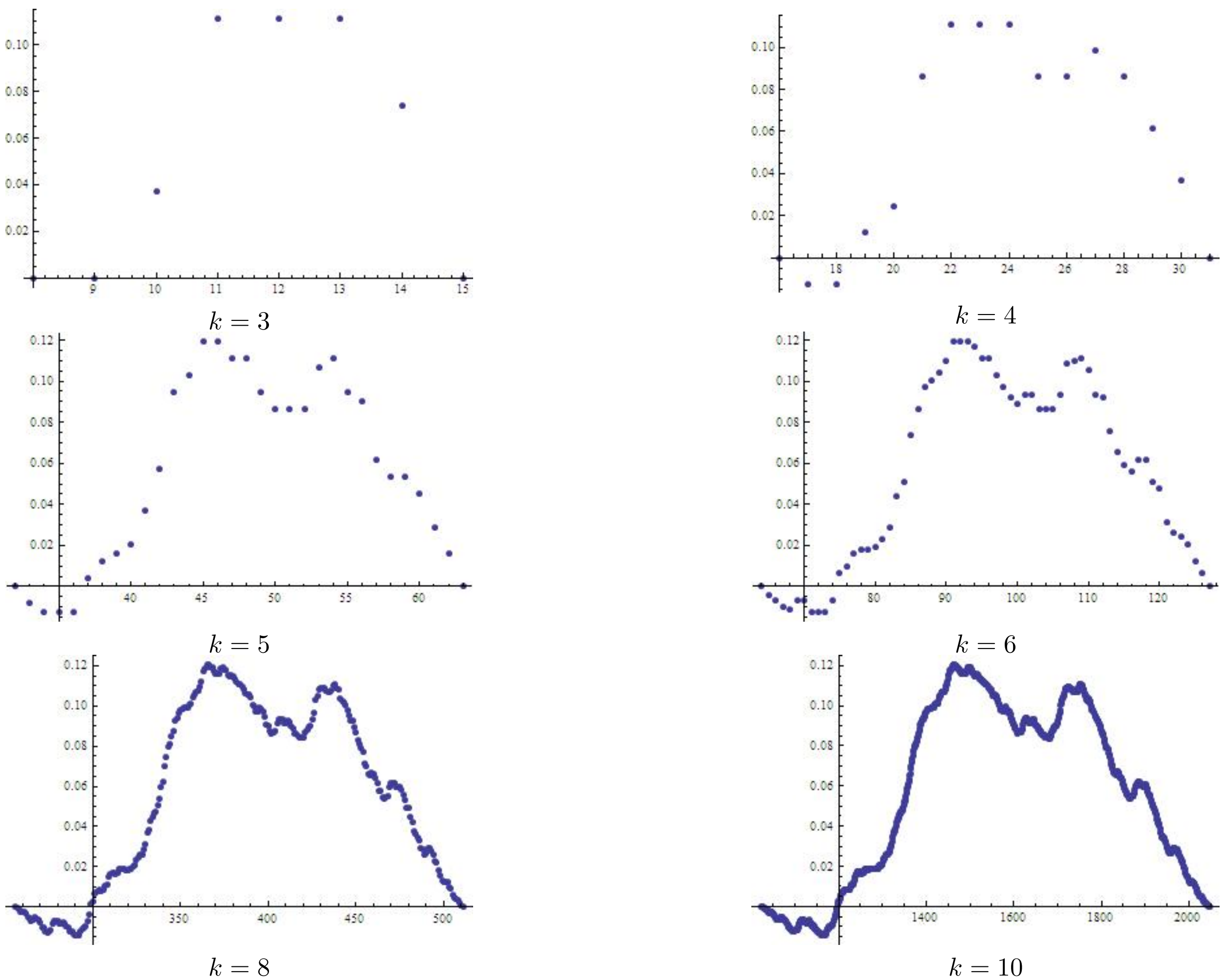


Figure 10.1: $\sum_{n=2^k}^{2^k(1+t)} {}^{*} (s(n) - w(n))$ for $k = 3, 4, 5, 8, 10$

us to make the following conjecture.

Conjecture 10.2.1. *There exists a continuous function* $\Phi(t)$ *on* $[0,1]$*, such that*

$$\lim_{k\to\infty} 3^{-k} \sum_{n=2^k}^{2^k(1+t)}{}^{*} (s(n) - w(n)) = \Phi(t).$$

Reznick [29] showed that for t in $[0,1]$, the function defined as

$$h(t) := \lim_{k\to\infty} 3^{-k} \sum_{n=2^k}^{2^k(1+t)}{}^{*} s(n),$$

extends to a continuous, strictly increasing function $H(t)$. This continuous extension $H(t)$ is not differentiable everywhere. The derivative $H'(t)$ exists and equals 0 for dyadic rationals, but $H'(t)$ does not exist for $t = m/(3 \cdot 2^r)$ with $\gcd(3,m) = 1$. Therefore, it is very likely that $\Phi(t)$ is not differentiable everywhere.

A few remaining questions are then, what is $\Phi(t)$ and how do you find it?

Chapter 11

Polynomial Analogue

The Stern Polynomial analogue, as defined by Klavžar, Milutinović, and Petr [22], satisfies the recurrences

$$\begin{gathered} S(0,x)=0, \qquad S(1,x)=1, \\ S(2n,x)=xS(n,x), \qquad S(2n+1,x)=S(n,x)+S(n+1,x). \end{gathered} \tag{11.1}$$

The first few polynomials are given as:

$$0,\ 1,\ x,\ 1+x,\ x^2,\ 1+2x,\ x(1+x),\ 1+x+x^2,\ x^3,\ 1+2x+x^2,\ldots.$$

Note that we recover the original Stern sequence by letting $x=1$, or in other words, $S(n,1)=s(n)$. The Stern polynomials also have the property $S(n,2)=n$. Also note that $(1+x)$ divides $S(3n,x)$. The generating function for $S(n,x)$ is

$$\sum_{n=0}^{\infty} S(n,x)t^n = t\prod_{n=0}^{\infty}\left(1+xt^{2^n}+t^{2^{n+1}}\right).$$

If we let $x=-1$, then we have a rephrasing of the product we saw in Chapter 5 in (5.3):

$$t\prod_{n=0}^{\infty}\left(1+-t^{2^n}+t^{2^{n+1}}\right)=t(1-t+t^3-t^4+t^6-t^7+t^9-\cdots)=t-t^2+t^4-t^5+t^7-t^8+t^{10}-\cdots.$$

This implies $S(3n,-1)=0$, and so equivalently, $1+x$ must divide $S(3n,x)$.

11.1 Definition

Since $1+x$ divides $S(3n,x)$ for all n, it is then natural to define the polynomial analogue of $w(n)$ to be

$$\hat{w}(n,x):=\frac{S(3n,x)}{1+x}.$$

Then the original sequence $w(n)$ is also recovered since $\hat{w}(n,1) = s(3n)/2 = w(n)$. Using the definition of this polynomial analogue, we generate the polynomials given in Table 11.1.

Table 11.1: Table for $\hat{w}(n,x)$

n	$\hat{w}(n,x)$	n	$\hat{w}(n,x)$	n	$\hat{w}(n,x)$
0	0	6	$x+x^2$	12	x^2+x^3
1	1	7	$1+3x$	13	$1+2x+2x^2$
2	x	8	x^3	14	$x+3x^2$
3	$1+x$	9	$1+2x+x^2$	15	$1+3x+2x^2$
4	x^2	10	$x+x^3$	16	x^4
5	$1+x^2$	11	$1+x+x^3$	17	$1+2x+3x^2$

11.2 Recurrences

Using (11.1), we have

$$\hat{w}(2^n,x) = \frac{S(3\cdot 2^n,x)}{(1+x)} = x^n \cdot \frac{S(3,x)}{(1+x)} = x^n.$$

More generally, we also have

$$\hat{w}(2^n\cdot k,x) = \frac{S(2^n\cdot 3k,x)}{(1+x)} = x^n \cdot \frac{S(3k,x)}{(1+x)} = x^n\cdot \hat{w}(k,x). \tag{11.2}$$

Using the definition for the polynomial analogue of $w(n)$, we have

$$\hat{w}(2n\pm 1,x) = \frac{S(3n\pm 1,x)+S(3n\pm 2,x)}{1+x}. \tag{11.3}$$

Similarly, we also find

$$\begin{aligned}\hat{w}(4n\pm 1,x) = \frac{S(12n\pm 3,x)}{1+x} &= \frac{S(3n,x)+S(3n\pm 1,x)+xS(3n\pm 1,x)}{1+x} \\ &= \hat{w}(n,x)+S(3n\pm 1,x),\end{aligned} \tag{11.4}$$

and

$$\begin{aligned}\hat{w}(4n\pm 3,x) = \frac{S(12n\pm 9,x)}{1+x} &= \frac{S(3n\pm 2,x)+xS(3n\pm 2,x)+S(3n\pm 3,x)}{1+x} \\ &= \hat{w}(n+1,x)+S(3n\pm 2,x).\end{aligned}$$

Using the definition for $\hat{w}(8n \pm 1, x)$ and (11.4), we have

$$\hat{w}(8n \pm 1, x) = \hat{w}(4n \pm 1, x) + 2x \cdot \hat{w}(n, x). \tag{11.5}$$

Rearranging (11.4) to obtain $S(3n \pm 1, x) = \hat{w}(4n \pm 1, x) - \hat{w}(n, x)$, and then substituting this in after using the definition in $\hat{w}(8n \pm 3, x)$, we have

$$\hat{w}(8n \pm 3, x) = x \cdot \hat{w}(4n \pm 1, x) + \hat{w}(2n \pm 1, x) - x \cdot \hat{w}(n, x). \tag{11.6}$$

To obtain recurrences for arithmetic progressions modulo 16, we replace n by $2n$ or $2n \pm 1$ in (11.5) and (11.6). We have

$$\begin{aligned}
\hat{w}(16n \pm 1, x) &= \hat{w}(8n \pm 1, x) + 2x^2 \cdot \hat{w}(n, x) \\
&= \hat{w}(4n \pm 1, x) + 2x(1 + x) \cdot \hat{w}(n, x), \\
\hat{w}(16n \pm 3, x) &= (x + 1)\hat{w}(4n \pm 1, x) + x^2 \cdot \hat{w}(n, x), && (11.7) \\
\hat{w}(16n \pm 5, x) &= (x^2 + 1)\hat{w}(4n \pm 1, x) + (x - 1)\hat{w}(2n \pm 1, x) - x^2 \cdot \hat{w}(n, x), && (11.8) \\
\hat{w}(16n \pm 7, x) &= \hat{w}(8n \pm 1, x) + 2x \cdot \hat{w}(2n \pm 1, x).
\end{aligned}$$

These recurrence relations are similar to the ones for the original sequence, and when $x = 1$, we recover the original recurrence relations for $w(n)$. We also define the following notation.

Definition 11.2.1. Let $G_k(x) := \sum_{j=1}^{k} x^j$.

We can generalize Theorem 7.1.1 to obtain the following theorem.

Theorem 11.2.2. *For all natural numbers n and $k > 2$,*

$$\hat{w}(2^k n \pm 1, x) = \hat{w}(2^{k-1} n \pm 1, x) + 2x^{k-2}\hat{w}(n, x) = \hat{w}(4n \pm 1, x) + 2\hat{w}(n, x)G_{k-2}(x).$$

Proof. In (11.5), replace n with $2^{k-3}n$ and then use (11.2) to obtain the first equality of the theorem. To obtain the second part, simply apply and reiterate the first equality of the theorem. □

In a similar manner we also obtain the following theorems.

Theorem 11.2.3. *For all natural numbers n and $k \geq 4$,*

$$\hat{w}(2^k n \pm 3, x) = (x+1)\hat{w}(2^{k-2}n \pm 1, x) + 2G_{k-4}(x) \cdot \hat{w}(n,x) + x^{k-2} + x^{k-2}\hat{w}(n,x) \tag{11.9}$$

$$= (x+1)\hat{w}(4n \pm 1, x) + 2(x+1)G_{k-4}(x)\hat{w}(n,x) + x^{k-2}\hat{w}(n,x). \tag{11.10}$$

Proof. In (11.7), replace n with $2^{k-4}n$ and then use Theorem 11.2.2 and (11.2) to obtain (11.9). To obtain (11.10), simply apply and reiterate Theorem 11.2.2. □

Theorem 11.2.4. *For all natural numbers n and $k \geq 5$,*

$$\hat{w}(2^k n \pm 5, x) = (x^2+1)\hat{w}(2^{k-2}n \pm 1, x) + (x-1)\hat{w}(2^{k-3}n \pm 1, x) - x^{k-2}\hat{w}(n,x) \tag{11.11}$$

$$= x(x+1)\hat{w}(4n \pm 1, x) + (2x(x+1)G_{k-5}(x) + 2(x^2+1)x^{k-4} - x^{k-2})\hat{w}(n,x). \tag{11.12}$$

Proof. In (11.8), replace n with $2^{k-4}n$ and then use Theorem 11.2.2 and (11.2) to obtain (11.11). To obtain (11.12), simply apply and reiterate Theorem 11.2.2. □

Ideally, we would like to generalize these recurrences and find a reduction formula for the polynomial analogue similar to Theorem 7.2.2.

11.3 Zeros of the Polynomial Analogues

We can consider $\hat{w}(n,x)$ as a sequence of polynomials. What are the zeros of this sequence of polynomials? First note, though, that since $\hat{w}(2^k n, x) = x^k \hat{w}(n,x)$ by (11.2), the zeros for these polynomials are the zeros of $\hat{w}(n,x)$ and $x = 0$ with multiplicity k. Thus, the even polynomials have been omitted from Table 11.2. We organize this information into a graph. Figures 11.1 and 11.2 show the zeros of $\hat{w}(n,x)$, for various values of n. The first graph shows the zeros of the polynomial analogue up to $n = 2^{10}$, while the second graph shows the zeros up to 2^{15}. These graphs are very similar to the graph of the zeros of $S(n,x)$. Figure 11.3 plots the zeros of $S(n,x)$ up to $n = 2^{11}$.

11.4 Other Properties

When looking at the table of polynomials, we see a few interesting patterns. First, when evaluated at $x = 0$, $\hat{w}(n,0)$ becomes a sequence of zeros and ones, more specifically, the repeated sequence $0, 1$. Also, similar to the Stern polynomials, we have $\hat{w}(n,2) = n$.

Table 11.2: Zeros of $\hat{w}(n,x)$

n	$\hat{w}(n,x)$	zeros
0	0	0
1	1	none
3	$x+1$	-1
5	x^2+1	$\pm i$
7	$3x+1$	$-1/3$
9	x^2+2x+1	$-1,-1$
11	x^3+x+1	$-0.682328,\ 0.341164 \pm 1.16154i$
13	$2x^2+2x+1$	$-0.5 \pm 0.5i$
15	$2x^2+3x+1$	$-1,-0.5$
17	$3x^2+2x+1$	$-0.33333 \pm 0.471405i$
19	x^3+2x^2+x+1	$-1.75488,\ -0.122561 \pm 0.744862i$
21	x^4+x^2+1	$\pm 0.5 \pm 0.866025i$
23	x^3+2x^2+3x+1	$-0.43016,\ -0.78492 \pm 1.30714i$
25	$4x^2+4x+1$	$-0.5,\ -0.5$
27	$2x^3+x^2+3x+1$	$-0.345627,\ -0.0771863 \pm 1.20029i$
29	$3x^2+4x+1$	$-1,\ -0.333333$
31	$2x^3+2x^2+3x+1$	$-0.396608,\ -0.301696 \pm 1.08151i$

Theorem 11.4.1. *For all natural numbers n, $\hat{w}(n,2)=n$.*

Proof. This follows directly from the property for the polynomial analogue for the Stern sequence:

$$\hat{w}(n,2)=\frac{S(3n,2)}{3}=\frac{3n}{3}=n. \qquad \square$$

It is interesting that $\hat{w}(n,x)$ can be written as a sum of previous polynomials, and the indices add up to n. For example, $\hat{w}(5,x)=\hat{w}(1,x)+\hat{w}(4,x)$ and $\hat{w}(9,x)=\hat{w}(6,x)+\hat{w}(3,x)$. For some n, there is more than one way to write it as a sum of previous polynomials. We can ask, in how many ways can each polynomial be written as a sum of the previous polynomials? If we associate each irreducible polynomial with its corresponding n value, then we can also think of these sums as partitions of n. This type of partition is given in Table 11.3. Can these partitions relate to $w(n)$ at all? Is there a known partition function which corresponds to counting these partitions of n?

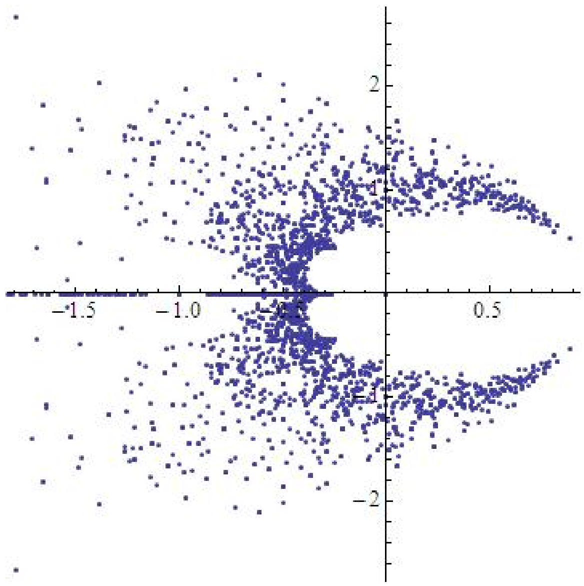


Figure 11.1: Zeros of $\hat{w}(n,x)$ up to $n = 2^{10}$

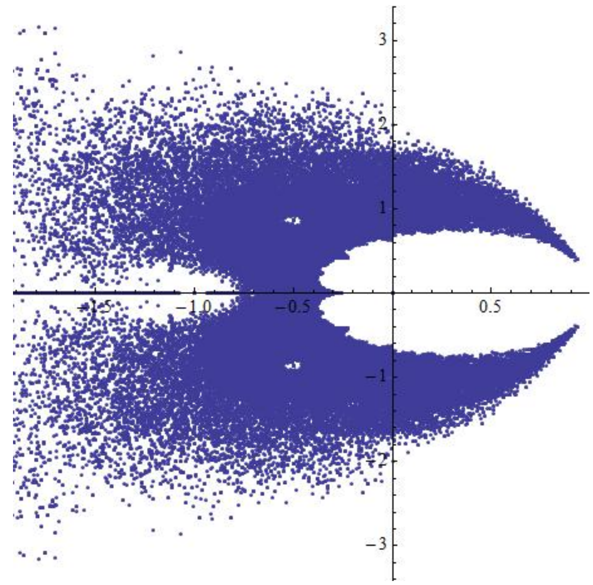


Figure 11.2: Zeros of $\hat{w}(n,x)$ up to $n = 2^{15}$

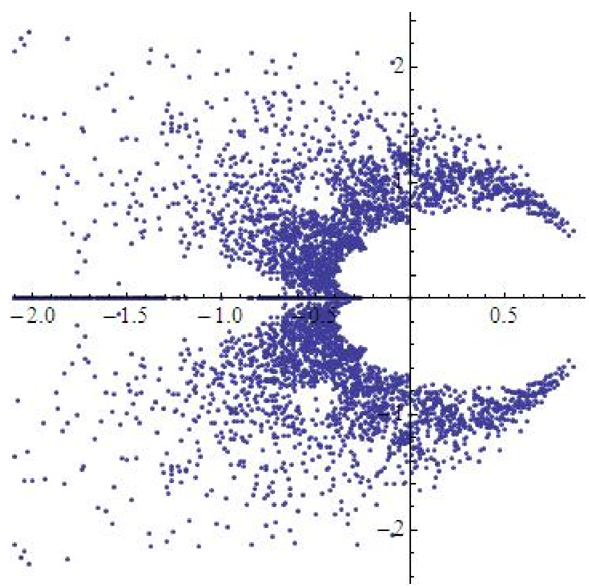


Figure 11.3: Zeros of $S(n,x)$ up to $n = 2^{11}$

Table 11.3: Ways to write $\hat{w}(n,x)$ as a sum of previous $\hat{w}(n,x)$

n	$\hat{w}(n,x)$	ways to write n	# of ways to write $\hat{w}(n,x)$
0	0	0	1
1	1	1	1
2	x	2	1
3	$x+1$	2+1	1
4	x^2	4	1
5	x^2+1	4+1	1
6	x^2+x	4+2	1
7	$3x+1$	3+2+2 2+2+2+1	2
8	x^3	8	1
9	x^2+2x+1	6+3 5+2+2 4+3+2 4+2+2+1	4
10	x^3+x	8+2	1
11	x^3+x+1	10+1 8+2+1	2
12	x^3+x^2	8+4	1
13	$2x^2+2x+1$	9+4 6+6+1 6+5+2 5+4+2+2 4+4+3+2 4+4+2+2+1	6
14	$3x^2+x$	6+4+4 4+4+4+2	2
15	$2x^2+3x+1$	13+2 9+4+2 7+4+4 6+6+3 6+6+2+1 6+5+2+2 6+4+2+2+1 5+4+2+2+2 4+4+2+2+2+1	9

Chapter 12

Future Directions

We highlight the open questions brought up in this thesis for future directions of research.

12.1 For the Stern Sequence

There are still many unanswered questions involving the Stern sequence, but we highlight questions introduced in this thesis.

We found recurrences and formulas for the second and third largest values in a row of the diatomic array. What are the formulas for the 4th largest value, or even the m-th largest value? We also made the conjecture that the m-th largest value satisfies the Fibonacci recurrence

$$L_m(r) = L_m(r-1) + L_m(r-2),$$

for all $r \geq 4m-2$. Is there an easy way to prove this? Proving the second and third largest values in a row satisfy a Fibonacci recurrence involved investigating numerous cases and seemed cumbersome. For a general case, the induction might be more complicated.

Related to this conjecture, we also discuss the possibility that another recurrence relation holds for $r \geq 4(m-1)$:

$$L_m(r) = L_{m-1} - F_{r-(4m-5)}.$$

If this is true, then this implies the gaps between values of the Stern sequence, when ranked according to size, are $F_{r-(4m-5)}$. This conjecture, if proved, would be useful in proving the conjecture that the normalized distribution of gaps is $\phi^{-(4k-3)}$ for $k \geq 2$.

Currently there is more known about the distribution of gaps for the Stern sequence than the distribution of values. The distribution of values seems to converge to a limiting function, but what is this function? How do we find it?

In Chapter 3, we discussed finding bounds for the row in which a value will appear in the diatomic array. If

$v(m)$ is the counting function which gives the first row in which m will appear and $f(x) = [\ln(\sqrt{5}x)/\ln\phi] - 2$, we conjectured that for all m,

$$f(m) \leq v(m) \leq f(m) + 3.$$

Again, this conjecture has a consequence for a bound on the smallest possible sum of continuants for a continued fraction, and this might prove useful in the area of continued fractions.

In addition to finding bounds for the first appearance of a value in a row of the diatomic array, what about finding asymptotics which give the second appearance of a value? Are there asymptotics that predict in which rows the new appearances of a value will occur?

12.2 For $w(n)$

There is still much to understand for $w(n)$. For example, a combinatorial interpretation for $w(n)$ independent of $s(n)$ is still unknown. The combinatorial interpretation in terms of $s(n)$ for $w(n)$ has a proof using the generating function, but what about a bijective proof? Connected to the combinatorial interpretation is the generating function. Once we find a combinatorial interpretation, will this lead to a nice closed formula for the generating function?

In Chapter 9, we discussed certain patterns for consecutive terms modulo 2. We could take this a step further. Is $w(n)$ eventually periodic modulo 2? If $w(n)$ is periodic modulo 2, then this will have implications for finding a combinatorial interpretation and the generating function.

We also considered the largest value of $w(n)$ in a row. What about the second largest value? These probably depend on the 4th, 5th, and 6th largest values of the Stern sequence, but is there an explicit formula for these values?

In Chapter 10, we considered the sum of $w(n)$ for n between powers of 2. But what is the sum of values across a row in the triangular array? We also conjecture there exists a continuous function $\Phi(t)$ on $[0, 1]$ such that

$$\lim_{k\to\infty} 3^{-k} \sum_{n=2^k}^{2^k(1+t)}{}^{*} (s(n) - w(n)) = \Phi(t).$$

How do we find this limiting function? We can also investigate sums of higher moments. For example, how does the sum of $(s(n) - w(n))^2$ over powers of 2 behave? Taking an initial glance, we expect each term in the expansion of $(s(n) - w(n))^2$ to satisfy a recurrence, since sums of $s(n)^2$ satisfy a recurrence. What is the recurrence relation for the entire sum?

The sequence $s(n)/s(n + 1)$ also provides an explicit enumeration of the positive rationals. While not investigated in this dissertation, we ask what values appear and do not appear in considering the sequence $w(n)/w(n+1)$?

12.3 Similar Sequences

We can construct sequences similar to $w(n)$ from the Stern sequence. For example, $s(5n)$ has the same parity as $s(n)$, and so the sequence given by $s(5n) \pm s(n)$ is always even. In fact, we can consider more general sequences of the form $s((2k+1)n) \pm s(n)$, and this sequence will always be even, as long as $2k+1$ is not a multiple of 3. Does this family of sequences have any interesting properties as well? Will these sequences have independent recurrences? Will they inherit any properties from the Stern sequence? How does this family of sequences behave?

References

[1] K. Anders, M. Dennison, J. Lansing, B. Reznick, *Congruence properties of binary partition functions*, Ann. Combin. **17** (2013) no. 1, 15–26.

[2] E. Alkan, M. Xiong, A. Zaharescu, *Pair correlation of sums of rationals with bounded height.* J. Reine Angew. Math. **641** (2010), 21–67.

[3] P. Bachmann, *Niedere Zahlentheorie*, (German) v. 1, Leipzig 1902. Reprinted by Chelsea, New York, 1968.

[4] A. Brocot, *Calcul des rouages par approximation, nouvelle méthode*, (French) Paris, 1862, 97 pp.

[5] N. Calkin and H. Wilf, *Recounting the rationals*, Amer. Math. Monthly **107** (2000), 360–363.

[6] L. Carlitz, *Single variable Bell polynomials*, Collect. Math. **14** (1962), 13–25.

[7] L. Carlitz, *A problem in partitions related to the Stirling numbers*, Bull. Amer. Math. Soc. **70** (1964), 275–278.

[8] L. Carlitz, *Some partition problems related to the Stirling numbers of the second kind*, Acta Arith. **10** (1964/1965), 409–422.

[9] G. de Rahm, *Un peu de mathématiques à propos d'une courbe plane*, Elem. Math. **2** (1947), 73–76, 89–97.

[10] L. E. Dickson, *History of the Theory of Numbers*, v. 1, Carnegie Inst. of Washington, Washington, D.C., 1919.

[11] E. W. Dijsktra, *Selected writings on computing: a personal perspective*, Springer-Verlag, New York, 1982.

[12] K. Dilcher and K. Stolarsky, *A polynomial analogue to the Stern sequence*, Int. J. Number Theory **3** (2007), no. 1, 85–103.

[13] K. Dilcher and K. Stolarsky, *Stern polynomials and double-limit continued fractions*, Acta Arith. **140** (2009), no. 2, 119–134.

[14] P. Gallagher, *On the distribution of primes in short intervals*, Mathematika **23** (1976), 4–9.

[15] R. Graham, M. Grötschel, and L. Lovász, *Handbook of Combinatorics*, Vol. 2, Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995.

[16] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics*, Second Edition, Addison-Wesley, Boston, MA, 1994.

[17] G. Halphen, *Sur des suites de fractions, analogues a la suite de Farey*, (French) Bull. Soc. Math. France. **5** (1877), 170–175.

[18] J. Hermes, *Anzahl der Zerlegungen einer ganzen rationalen Zahl in Summanden*, (German) Math. Ann. **45** (1894), no. 3, 371–380.

[19] C. Hooley, *An asymptotic formula in the theory of numbers*, Proc. London Math. Soc. **7** (1957), 396–413.

[20] C. Hooley, *On the intervals between consecutive terms of sequences*, Proc. Symp. Pure Math. **24** (1973), 129–140.

[21] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1992.

[22] S. Klavžar, U. Milutinović, and C. Petr, *Stern polynomials*, Adv. in Appl. Math. **39** (2007) no. 1, 86–95.

[23] J. Lansing, *Distribution of values of the binomial coefficients and the Stern sequence*, J. Integer Seq. **16**, no. 3, (2013), Article 13.3.7, 10 pp.

[24] D. H. Lehmer, *On Stern's diatomic series*, Amer. Math. Monthly, **36** (1929), 59–67.

[25] D. A. Lind, *An extension of Stern's diatomic series*, Duke Math. J. **36** (1969), 55–60.

[26] E. Lucas, *Sur les suites de Farey*, (French) Bull. Soc. Math. France, **6** (1878), 118–119.

[27] B. Reznick, *Regularity Properties of the Stern Enumeration of the Rationals*, J. Integer. Seq. **11** (2008), no. 4, Article 08.4.1, 17 pp.

[28] B. Reznick, *Some binary partition functions*, Analytic number theory (Allerton Park, IL, 1989), 451–477, Progr. Math., 85, Birkhaüser Boston, Boston, MA, 1990.

[29] B. Reznick, *A Stern introduction to combinatorial number theory*, in preparation.

[30] Z. Rudnick, A. Zaharescu, *The distribution of spacings between fractional parts of lacunary sequences*, Forum Math. **14** (2002), no. 5, 691–712.

[31] E. Seneta, *Non-Negative Matrices*, Halsted Press, New York, 1973.

[32] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*. Published electronically at `http://oeis.org`.

[33] V. Sós, *On the distribution mod 1 of the sequence* $n\alpha$, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. **1** (1958), 127–134.

[34] M. A. Stern, *Über eine zahlentheoretische Funktion*, (German) J. Reine Angew. Math. **55** (1858), 193–220.

[35] P. Stănică, *Good lower and upper bounds on binomial coefficients*, J. Inequal. Pure Appl. Math. **2** (2001), no. 3, Article 30, 5 pp.

[36] S. Swierczkowski, *On successive settings of an arc on the circumference of a circle*, Fundam. Math. **46** (1958), 187–189.

[37] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, (German) Math. Annal. **77** (1916), 313–352.