Numbers n with Two Regions in the Symmetric Representation of sigma(n)

Hartmut F. W. Höft
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All references to Lemmas and Theorems are in the paper referenced in the LINK section of A241561.

LEMMA A:

Let n = q×p with q ∊ A174973, p prime and 2×q < p then $c_n = 2$, i.e., n ∊ A239929.

PROOF:

Lemma 1(d) implies that r_n < p so that all odd divisors d of q satisfy d < r_n. These are represented as 1’s in odd-numbered positions of the n-th row of the irregular triangle of A237048. There is an equal number of odd divisors of n that have p as a factor. These are represented as 1’s in even-numbered positions of the n-th row of the irregular triangle of A237048.

Now, let q = 2^m×s with m ≥ 0 with s odd. By Lemma 1(e) divisor s×p as the largest odd divisor of n is represented as a 1 in column 2^{m+1} of the irregular triangle A237048, and divisor p > r_n as a 1 in column 2^{m+1}×s = 2×q ≤ r_n.

If s = 1 then the only 1's occur in positions 1 and $2^{m+1} \le r_n$, i.e., there is only one region of width 1 that ends at leg 2^{m+1} together with its symmetric copy.

If s > 1 and d > 1 its smallest odd divisor d > 1 then d < 2^{m+1}, since 2^m is a divisor of n and 2 × 2^m < d violates the assumption that q is in A174973. Let 1 = $d_1 < \dots < d_k$ = s, for some k > 1, be all odd divisors of s. Then $d_{j+1} < 2^{m+1} \times d_j$, for all 1 ≤ j < k. Therefore, for any position 1 ≤ i < $2^{m+1} \times d_k$ the number of 1’s in odd-numbered positions before position i in the n-th row of the irregular triangle A237048 is strictly larger than the number of 1’s in even-numbered positions and these two counts become equal only at position $2^{m+1} \times d_k$. Since there are no further 1’s beyond position $2^{m+1} \times d_k$ there is exactly one region before the center of the Dyck path together with its symmetric copy.

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LEMMA B:

If $c_n = 2$, i.e. n ∊ A239929, then n = q×p with q ∊ A174973, p prime and 2×q < p.

PROOF:

By Lemma 4, number n has no middle divisors. If we let n = $2^m \times s$ > 1 where m ≥ 0 and s is odd, then s > 1 by the proof of Conjecture #1 in the LINKS section of A238443. Also, there are odd numbers d, e ∊ ℕ such that s = d×e, $2^m \times d$ is the largest divisor of n less than r_n, and e is the smallest divisor of n greater than r_n. Therefore, $2^m \times d \le \sqrt{n/2} < r_n < \sqrt{2n}$ < e, and 2 × ($2^m \times d$) < e. In addition, the number of odd divisors of s less than r_n is equal to the number of odd divisors of s greater than r_n. Finally, let 1 = $d_1 < \dots < d_k$, for some k > 1, be all divisors of s.

Suppose m = 0. Then in the n-th row of the irregular triangle of A237048 position 1, representing divisor 1, as well as position 2, representing divisor n = s, contain 1’s so that c_n > 2 because position d_2 > 2 contains a 1. Therefore, m > 0.

Suppose that for some 1 ≤ j < k, $d_j < 2 \times d_j < d_{j+1}$ holds and that d_{j+1} is even, say d_{j+1} = 2×u. Then $d_j < 2 \times d_j$ < 2×u, i.e., d_j < u < 2×u = d_{j+1} contradicting the assumption that d_j and d_{j+1} are successive divisors of n. Suppose that $d_i = 2^h \times$v for 1 ≤ h < m and odd v, then 2×d_i is a divisor contradicting the assumption that d_i and d_{i+1} are successive divisors of n. Therefore, $d_i = 2^m \times$v. Let 1 ≤ i < k be the smallest index such that $d_i < 2 \times d_i < d_{i+1}$ holds. Consider all odd divisors less than or equal to v, say $1 = e_1 < \ldots < e_\alpha = v$. Since $2^{m+1} \times v < d_{i+1}$, the number of 1's in the odd positions $e_1 < \ldots < e_\alpha$ equals the number of 1's in the even positions $2^{m+1} \times e_1 < \ldots < 2^{m+1} \times e_\alpha$ in the n-th row of the irregular triangle A237048. In other words, at least one region ends at leg $2^{m+1} \times e_\alpha$. However, since there is a 1 in position $d_{i+1} > 2^{m+1} \times e_\alpha$, $c_n > 2$ contradicting the assumption. Therefore, $d_i < 2 \times d_i < d_{i+1}$ cannot occur for any index and factor $2^m \times d \in$ A174973.

Suppose that $e = x \times y$ with $1 < x, y$. Then $2^m \times d < 2^m \times d \times x < 2^m \times d \times y < r_n$ since e is the smallest divisor of n larger than r_n contradicting that $2^m \times d$ is the largest divisor of n less than r_n. Therefore, e is prime.

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These two Lemmas establish the following equivalencies.

THEOREM:

For every number $n \in \mathbb{N}$:
$c_n = 2 \iff n \in$ A239929 $\iff n = q \times p$, where $q \in$ A174973 and p is a prime satisfying $2 \times q < p$.