Materials in support of A231327 (denominator) and A231273 (numerator)
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Integer Denominators and Integer Components of the Numerators of a Close Variant of the Infinite Product Involving All, and Only, the Primes in Euler's So-called Product Formula

by Leo Depuydt

Short Statement of Purpose

The starting focus of the two proposed sequences is on the infinite prime product of Euler's so-called product formula as considered for even exponents only, namely

$$\frac{2^{2n} \times 3^{2n} \times 5^{2n} \times 7^{2n} \times 11^{2n} \times \ldots}{(2^{2n}-1)(3^{2n}-1)(5^{2n}-1)(7^{2n}-1)(11^{2n}-1) \times \ldots}$$

The main design of the proposal is to provide the numerical values of a close variant, the one in which all "–" is simply changed into "+."

$$\frac{2^{2n} \times 3^{2n} \times 5^{2n} \times 7^{2n} \times 11^{2n} \times \ldots}{(2^{2n}+1)(3^{2n}+1)(5^{2n}+1)(7^{2n}+1)(11^{2n}+1) \times \ldots}$$

Considering the role of Euler's product formula in the study of the prime numbers, the numerical values of the product with "+" may be of interest even though it has been derived from the product sequence with "–." Though one is derived from the other, they are in some sense equivalent. There is *a priori* no compelling reason on account of which the product with "–" should be more telling about the true nature of the prime numbers than the product with "+."

The numerical values of the product with "+" must have been known to some in the past. But I have not been able to find them explicitly mentioned anywhere.

I also note that I cannot account for one number in sequence A0446988 in www.oeis.org (see below). Is a correction in order?

Introduction

The proof that follows below is fairly simple and it constitutes a shorter version of the proof needed for A114362 and A114363. I obtained the results for A114362 and A114363 independently, as the proof provided below shows, since no proof is after all given for the entries A114362 and A114363 of www.oeis.org. The search for confirmation of my own results is what led me to the site www.oeis.org for the first time.

What follows is a variation on the same theme as A114362 and A114363. What may be intriguing about the variation is that it presents the infinite prime product of Euler's celebrated product formula with only a *minimal* change in form, namely of all "–" into "+". Infinite series involving the primes are hard to come by, so it may be useful to look at the ones that we do have from every possible angle.

The result with "+" is derivative of the result with "–." I wonder whether it will be possible to derive the result for "+" independently from the result with "–." Publishing the result for "+" may encourage someone out there to make the attempt. It may not be possible.

The Two Sequences in Question and Two Interesting Properties Exhibited by Them

Denominator (9 terms)

15, 105, 675675, 34459425, 16368226875, 218517792968475, 30951416768146875, 694097901592400930625, 23383376494609715287281703125

Integer component of numerator (9 terms)

1, 1, 691, 3617, 174611, 236364091, 3392780147, 7709321041217, 26315271553053477373

An Interesting Property of the Rational Components of the Numerators

This sequence is found elsewhere. It has to do with the Bernoulli numbers. One finds it in entries A046988, A001067, A046968, A141590, A098087, and A156036 of www.oeis.org. But there is doubling, or alternation of plus and minus, or separation by other numbers, and so on. Here, the sequence appears purely as is.

In this connection, an observation on A046988 is in order. The 15th number is 6,785,560,294. But that is double of what is found in the present series. I note that zeta(14) has 2 times Pi^14 in the numerator. Is this "2" perhaps related to the divergence?

An Interesting Property of the Denominators

To a great degree, the earlier members of the sequences tend to be divisors of the later members. As far as the numbers at my disposition are concerned, I provisionally observe a higher degree of this phenomenon with the "+" prime product than with "–" product. For example, 15, 105, 675675, and 34459425 are all divisors of all the higher numbers in the sequence of nine that I have computed. The first seven numbers are all divisors of the ninth. But all this might change with every higher numbers. It would be interesting to know whether the conjecture that all numbers of the sequence have the first number as a divisor. Perhaps, it might be worthwhile to investigate with computer computing of ever higher numbers the provisional conjecture that there is a higher rate of divisibility of what follows by what precedes in the "+" product than in the "–" product.

Information for the Most Part as Initially Entered on the Website www.oeis.org

Entry 1

NAME: Denominator of a close variant of Euler's infinite prime product, i.e. prod((p^{2n}/(p^{2n}–1), with all "–" changed into "+," as follows: (p^{2n})/(p^{2n}+1).

15, 105, 675675, 34459425, 16368226875, 218517792968475, 30951416768146875, 694097901592400930625, 23383376494609715287281703125

FORMULA: prod((p^{2n})/(p^{2n}+1)=zeta(4n)/zeta(2n), the product being over all the primes.

CROSSREFS: Cf. A114362, A114363, and A231273; cf. also, for the corresponding numerator, A001067, A046968, A046988, A098087, A141590, and A156036.

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Entry 2

NAME: Integer component of the numerator of a close variant of Euler's infinite prime product, i.e. prod((p^{2n}/(p^{2n}–1), with all "–" changed into "+," as follows: (p^{2n})/(p^{2n}+1), the transcendental component being Pi^{2n}

1, 1, 691, 3617, 174611, 236364091, 3392780147, 7709321041217, 26315271553053477373

FORMULA: prod((p^{2n})/(p^{2n}+1)=zeta(4n)/zeta(2n), the product being over all the primes

CROSSREFS: Cf. A114362, A114363, A231327; cf. also A046988, A001067, A046968, A141590, A098087.

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Proof

The expression

$$\frac{2^4 \times 3^4 \times 5^4 \times 7^4 \times 11^4 \times \ldots}{(2^4-1)(3^4-1)(5^4-1)(7^4-1)(11^4-1) \times \ldots} \qquad (1)$$

is one manifestation of the prime product in Euler's product formula when the exponents are even, that is

$$\frac{2^{2n} \times 3^{2n} \times 5^{2n} \times 7^{2n} \times 11^{2n} \times \ldots}{(2^{2n}-1)(3^{2n}-1)(5^{2n}-1)(7^{2n}-1)(11^{2n}-1) \times \ldots}. \qquad (2)$$

Expression (1) evidently equals

$$\frac{2^4 \times 3^4 \times 5^4 \times 7^4 \times 11^4 \times \ldots}{(2^2-1)(2^2+1)(3^2-1)(3^2+1)(5^2-1)(5^2+1)(7^2-1)(7^2+1)(11^2-1)(11^2+1) \times \ldots}. \qquad (3)$$

Another manifestation of (2) is

$$\frac{2^2 \times 3^2 \times 5^2 \times 7^2 \times 11^2 \times \ldots}{(2^2-1)(3^2-1)(5^2-1)(7^2-1)(11^2-1) \times \ldots}. \qquad (4)$$

If (3) be divided by (4), then the result after simplification is

$$\frac{2^2 \times 3^2 \times 5^2 \times 7^2 \times 11^2 \times \ldots}{(2^2+1)(3^2+1)(5^2+1)(7^2+1)(11^2+1) \times \ldots}. \qquad (5)$$

Incidentally, the results for A114362 and A114363 are obtained by dividing (5) once more by (4). I intend to discuss elsewhere the fact that these results are rational. I hope to suggest that the result for the prime product goes deeper than A114362 and A114363.

It is known through Euler’s product formula that expression (1), and therefore also expression (3), equals zeta(4) = $\frac{\pi^2}{90}$ and also that expression (4) equals zeta(2) = $\frac{\pi^2}{6}$.

Dividing (3) by (4), with (5) as the result, is therefore the same as dividing $\frac{\pi^2}{90}$ by $\frac{\pi^2}{6}$. The result is evidently $\frac{\pi^2}{15}$.

The same applies to the relation between

$$\frac{2^{2n} \times 3^{2n} \times 5^{2n} \times 7^{2n} \times 11^{2n} \times \ldots}{(2^{2n}-1)(3^{2n}-1)(5^{2n}-1)(7^{2n}-1)(11^{2n}-1) \times \ldots}$$

and

$$\frac{2^{4n} \times 3^{4n} \times 5^{4n} \times 7^{4n} \times 11^{4n} \times \ldots}{(2^{4n}-1)(3^{4n}-1)(5^{4n}-1)(7^{4n}-1)(11^{4n}-1) \times \ldots}$$

for all *n*.

All the numerators and denominators of the proposed sequences are therefore obtained by dividing zeta(4*n*) by zeta(2*n*). *QED.*

Relevant Equations Formulated Explicitly

$$\frac{2^2 \times 3^2 \times 5^2 \times 7^2 \times 11^2 \times \ldots}{(2^2+1)(3^2+1)(5^2+1)(7^2+1)(11^2+1) \times \ldots} = \frac{\pi^2}{15}$$

$$\frac{2^4 \times 3^4 \times 5^4 \times 7^4 \times 11^4 \times \ldots}{(2^4+1)(3^4+1)(5^4+1)(7^4+1)(11^4+1) \times \ldots} = \frac{\pi^4}{105}$$

$$\frac{2^6 \times 3^6 \times 5^6 \times 7^6 \times 11^6 \times \ldots}{(2^6+1)(3^6+1)(5^6+1)(7^6+1)(11^6+1) \times \ldots} = \frac{691\pi^6}{675{,}675}$$

$$\frac{2^8 \times 3^8 \times 5^8 \times 7^8 \times 11^8 \times \ldots}{(2^8+1)(3^8+1)(5^8+1)(7^8+1)(11^8+1) \times \ldots} = \frac{3617\pi^8}{34{,}459{,}425}$$

$$\frac{2^{10} \times 3^{10} \times 5^{10} \times 7^{10} \times 11^{10} \times \ldots}{(2^{10}+1)(3^{10}+1)(5^{10}+1)(7^{10}+1)(11^{10}+1) \times \ldots} = \frac{174{,}611\pi^{10}}{16{,}368{,}226{,}875}$$

$$\frac{2^{12} \times 3^{12} \times 5^{12} \times 7^{12} \times 11^{12} \times \ldots}{(2^{12}+1)(3^{12}+1)(5^{12}+1)(7^{12}+1)(11^{12}+1) \times \ldots} = \frac{236{,}364{,}091\pi^{12}}{218{,}517{,}792{,}968{,}475}$$

$$\frac{2^{14} \times 3^{14} \times 5^{14} \times 7^{14} \times 11^{14} \times \ldots}{(2^{14}+1)(3^{14}+1)(5^{14}+1)(7^{14}+1)(11^{14}+1) \times \ldots} = \frac{3{,}392{,}780{,}147\pi^{14}}{30{,}951{,}416{,}768{,}146{,}875}$$

$$\frac{2^{16} \times 3^{16} \times 5^{16} \times 7^{16} \times 11^{16} \times \ldots}{(2^{16}+1)(3^{16}+1)(5^{16}+1)(7^{16}+1)(11^{16}+1) \times \ldots} = \frac{7{,}709{,}321{,}041{,}217\pi^{16}}{694{,}097{,}901{,}592{,}400{,}930{,}625}$$

$$\frac{2^{18} \times 3^{18} \times 5^{18} \times 7^{18} \times 11^{18} \times \ldots}{(2^{18}+1)(3^{18}+1)(5^{18}+1)(7^{18}+1)(11^{18}+1) \times \ldots} = \frac{26{,}315{,}271{,}553{,}053{,}477{,}373\pi^{18}}{23{,}383{,}376{,}494{,}609{,}715{,}287{,}281{,}703{,}125}$$

And so on.