Filling a subtractive triangle
Minimal solutions

Complete research without symmetric solutions
(Results from Rec.pas and Rec_Dec.pas)

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H = 2

```
2   3
  1

1   3
  2
```

2 solutions in 800 µs for the maximum N = 3 / 3 terms.

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H = 3

```
5   2   6
  3   4
    1

4   1   6
  3   5
    2

5   6   2
  1   4
    3

4   6   1
  2   5
    3
```

4 solutions in 1.8 ms for the maximum N = 6 / 6 terms.

****

H = 4

```
8   1  10   6
  7   9   4
    2   5
      3

6   1  10   8
  5   9   2
    4   7
      3

9   3  10   8
  6   7   2
    1   5
      4

8   3  10   9
  5   7   1
    2   6
      4
```

4 solutions in 4.2 ms for the maximum N = 10 / 10 terms.

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```
H = 5                                 6  14  15   3  13
                                        8   1  12  10
                                          7  11   2
                                            4   9
                                              5
```

1 solution in 11.2 ms for the maximum N = 15 / 15 terms.

****

```
H = 6                              13  21   3  22  20   6
                                      8  18  19   2  14
                                       10   1  17  12
                                          9  16   5
                                            7  11
                                              4
```

1 solution in 32 ms for the maximum N = 22 / no solution in 0.020 s for the maximum N = 21 / 21 terms.

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```
H = 7                            19  32   3  33  31   6  17
                                   13  29  30   2  25  11
                                     16   1  28  23  14
                                       15  27   5   9
                                         12  22   4
                                           10  18
                                              8


                                 14  31   5  33  32   8  19
                                   17  26  28   1  24  11
                                      9   2  27  23  13
                                        7  25   4  10
                                         18  21   6
                                            3  15
                                             12
```

2 solutions en 98 ms for the maximum N = 33 / no solution in 30 ms for the maximum N = 32 / 28 terms.

****

```
H = 8                          29   6  43  44   3  42  33   7
                                 23  37   1  41  39   9  26
                                   14  36  40   2  30  17
                                     22   4  38  28  13
                                       18  34  10  15
                                         16  24   5
                                            8  19
                                             11
```

1 solution in 1.079 s for the maximum N = 44 / no solution in 0.589 s for the maximum N = 43 / 36 terms.

****

```
H = 9                        17  49  58   1  55  59  13  53  43
                               32   9  57  54   4  46  40  10
                                 23  48   3  50  42   6  30
                                   25  45  47   8  36  24
                                     20   2  39  28  12
                                       18  37  11  16
                                         19  26   5
                                            7  21
                                             14
```

1 solution in 40.54 s (1.3 10^{11} operations) for the maximum N = 59 / no solution in 27.42 s for the maximum N = 58 / 45 terms / 4.6 10^{15} possible arrangements on the last line.

****

H = 10

Minimal solution	Line number
45 69 12 76 73 15 75 66 16 52	10
24 57 64 3 58 60 9 50 36	9
33 7 61 55 2 51 41 14	8
26 54 6 53 49 10 27	7
28 48 47 4 39 17	6
20 1 43 35 22	5
19 42 8 13	4
23 34 5	3
11 29	2
18	1

1 solution in 1251 s (4.1 10^{12} operations) for the maximum N = 76 / no solution in 1.454 s for the maximum N = 75 / 55 terms / 3.5 10^{18} possible arrangements on the last line.

Number of try on each line

1	2	3	4	5	6	7	8	9	10	
1	1204	69866	4624615	189354752	3236467522	13721003558	9370916499	600869199	1380435	
0.0	0.0	0.0	0.0	0.7	11.9	50.6	34.5	2.2	0.0	%

Mean duration of one try : 53.6 ns S = 27124687651 ≈ 2.7 10^{10} t δ = 152 ops

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H = 11

32 73 91 11 99 101 6 100 96 22 75
41 18 80 88 2 95 94 4 74 53
23 62 8 86 93 1 90 70 21
39 54 78 7 92 89 20 49
15 24 71 85 3 69 29
9 47 14 82 66 40
38 33 68 16 26
5 35 52 10
30 17 42
13 25
12

64 16 97 101 11 93 99 19 92 79 21
48 81 4 90 82 6 80 73 13 58
33 77 86 8 76 74 7 60 45
44 9 78 68 2 67 53 15
35 69 10 66 65 14 38
34 59 56 1 51 24
25 3 55 50 27
22 52 5 23
30 47 18
17 29
12

67 14 101 91 9 94 100 16 96 81 22
53 87 10 82 85 6 84 80 15 59
34 77 72 3 79 78 4 65 44
43 5 69 76 1 74 61 21
38 64 7 75 73 13 40
26 57 68 2 60 27
31 11 66 58 33
20 55 8 25
35 47 17
12 30
18

3 solutions in 158441 s (5.2 10^{14} operations) for the maximum N = 101 / no solution en 795 s for the maximum N = 100 / 66 terms / 6.3 10^{21} possible arrangements on the last line.

****

H = 12

```
62  90  13 115 124   8 122 125   5 118 101  20
  28  77 102   9 116 114   3 120 113  17  81
    49  25  93 107   2 111 117   7  96  64
      24  68  14 105 109   6 110  89  32
        44  54  91   4 103 104  21  57
          10  37  87  99   1  83  36
            27  50  12  98  82  47
              23  38  86  16  35
                15  48  70  19
                  33  22  51
                    11  29
                      18
```

1 solution in 7140416 s (2.4 10^{16} operations) for the maximum N = 125 / no solution in 9404 s for the maximum N = 124 / 78 terms / 8.4 10^{24} possible arrangements on the last line.

****

H = 13

```
103  32 131 138   8 157 147   2 153 158  25 143 108
   71  99   7 130 149  10 145 151   5 133 118  35
     28  92 123  19 139 135   6 146 128  15  83
       64  31 104 120   4 129 140  18 113  68
         33  73  16 116 125  11 122  95  45
           40  57 100   9 114 111  27  50
             17  43  91 105   3  84  23
               26  48  14 102  81  61
                 22  34  88  21  20
                   12  54  67   1
                     42  13  66
                       29  53
                         24
```

```
116 139  22 158 151   4 157 154  21 156 143  25 115
   23 117 136   7 147 153   3 133 135  13 118  90
     94  19 129 140   6 150 130   2 122 105  28
       75 110  11 134 144  20 128 120  17  77
         35  99 123  10 124 108   8 103  60
           64  24 113 114  16 100  95  43
             40  89   1  98  84   5  52
               49  88  97  14  79  47
                 39   9  83  65  32
                   30  74  18  33
                     44  56  15
                       12  41
                         29
```

2 solutions in 8985 days (about 2.7 10^{18} operations) for the maximum N = 158 / no solution in 156095 + 458443 = 614538 s ≈ 7 days for the maximum N = 157 / 91 terms / 2.3 10^{28} possible arrangements on the last line.

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Remarks

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Algorithm

Used settings are the high of triangle H, the line number L and the column number C of an element. Number of elements in a line L is equal to L, which varies from 1 to H.

```
X   Y          L

  Z            L – 1
```

In the subtractive triangle $Z = |X - Y|$, so $Z = X - Y$ or else $Z = Y - X$. So $Y = X - Z$ or else $Y = X + Z$.

The exploration is made for values less or equal to maximum N considered. The number of line L is incremented when the line is completed. The number of column C is incremented to complete a line.

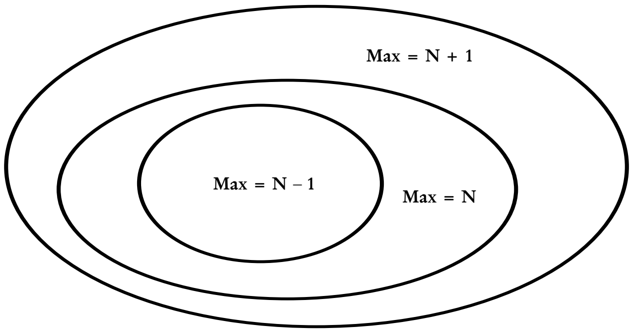
The program choose the first element of the line L to feel in the list of available elements, then calculate all the next elements realising subtractions or additions. Every first elements are tested, every possible additions and subtractions are realised. Complete course is realised with backtracking. The program step back when there is a failure (Y value not available or too big).

After the value of X and Z, one insert an available Y (a value not yet inserted, and which is not too big – see later).
Likewise $X = Y + Z$ or else $Y = X + Z$, so X or else Y is bigger then Z. For the elements to place on the next line, the inserted value Y (after the choice of X) doesn't be too big. A minimal sum S (calculated with still available elements) is to be added to Y. This minimal sum to place on next lines $(N - S)$ maximise the Y value to insert.

For all value H of subtractive triangle, it exist some solutions, and then a solution with a maximal value N minimal.
For example, a possible solution for $H = 5$, with a maximum $N = 103$:
a) the first term on each line is bigger than the maximum of the previous line,
b) on each line, we only use additions.

```
30    40    54    74   103
   10    14    20    29
      4     6     9
         2     3
            1
```

That is N the bigger term in a triangle of high H. That is $\mathscr{E}_{H,N}$ the set of solutions with maximum less or equal to N. The sets of solutions $\mathscr{E}_{H,N}$ are nested, one element of $\mathscr{E}_{H,N}$ is also an element of $\mathscr{E}_{H,N+1}$. Also, the absence of solution of rank N ($\mathscr{E}_{H,N}$ empty) induce the absence of solution of rank $N-1$ ($\mathscr{E}_{H,N-1}$ is also empty).



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Calculus / Researches

The symmetric solutions are not counted (left - right symmetry).

After value H = 9, verification of absence of solution is made with Rec_Dec.pas (*récursif découpé*). The first element on the first line is chosen, the first element on the second line is in the specified interval (if the option *intervalle* is active). This allow the decomposition of problem in lot of small tasks.

For H = 12, The case first line equal to 1 is about 1.9 % of calculus time.
For H = 13, The case first line equal to 1 is about 2.2 % of calculus time.

The execution times are normalised for a Xeon W3670 3.20-3.46 GHz (processors Laurent & Pierre). The incertitude on measurements is about ± 0.5 %. The incertitude on the time of a task is about ± 3 %. I used Turbo Pascal 7 to realise the calculus under Windows XP sp3 32 bits (the program is about 5 times faster than the Ocaml program using the same algorithm under Windows 7 (or Windows 10) 64 bits). The collection of results on distinct processors was made with software Anydesk 4.3.0 (free and compatible with Windows XP).

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The calculus realised for H = 13 take about 8985 days (25 years / About 242 days with 37.2 normalised processors / After 340 days of effective activity with 39 processors / Counting data lost, untimely restart, net failure, algorithm errors, different efficiency of processors, time-out, multi-threading Windows, repairs ...).

For N = 158 (supposed minimal value), the maximal value on first line is 80. The program divided the research in $30 + 30 + 50 \times 49 = 2510$ independent tasks. Duration of tasks vary from 10 milliseconds to 16 days (normalised). Efficiency of processors vary from 0.69 to 1.15 (Xeon X3360 to Xeon X5687), compared to Xeon W3670.

Calculus realised for H = 14 would have the duration of about 10^7 days (27000 years / 700 years with 39 processors / 3 years with 10000 processors).

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