The Rectangular Spiral Solution for the $n_1 X n_2 X \ldots X n_k$ Points Problem

Marco Ripà

Economics − Institutions and Finance, Roma Tre University, Rome, Italy.
Email: marcokrt1984@yahoo.it

Abstract. A generalization of Ripà's square spiral solution for the n X n X … X n points upper bound problem. Additionally, we provide a non-trivial lower bound for the k-dimensional n_1 X n_2 X … X n_k points problem. In this way, we can build a range in which certainly will fall all the best possible solutions to the problem we are considering. Finally, we provide a few characteristic numerical examples in order to appreciate the goodness of the result arising from the particular approach we have chosen.

Keywords: dots, straight line, inside the box, outside the box, plane, upper bound, lower bound, graph theory, segment, points.

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1. Introduction

About one century ago, the classic *nine dots problem* appears on Samuel Loyd's *Cyclopedia of Puzzles* [1-4]. The question was as follows: "Draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes" [3-5].

Little Tommy Riddles calls attention to a couple of Christopher Columbus' famous egg tricks. In the first puzzle the famous trick-chicken, Americus Vespucius, after whom our great country was named, showed a clever puzzle wherein you are asked to lay nine eggs so as to form the greatest possible number of rows of three-in-line. King Puzzlepate has only succeeded in getting eight rows, as shown in the picture, but Tommy says a smart chicken can do better than that! The funny old King is now trying to work out a second puzzle, which is to draw a continuous line through the center of all of the eggs so as to mark them off in the fewest number of strokes. King Puzzlepate performs the feat in six strokes, but from Tommy's expression we take it to be a very stupid answer, so we expect our clever puzzlists to do better; it is a very ingenious trick, fully as good if not better than that of making an egg stand up on end, for the perpetration of which with an over ripe egg the great navigator was loaded with chains.

Fig. 1. The original problem from Samuel Loyd's *Cyclopedia of Puzzles*, New York, 1914, p. 301.

That puzzle can be naturally extended to an arbitrary large number of distinct (zero-dimensional) points for each raw / column [7]. This new problem asks to connect n X n points, arranged in a grid formed by n rows and n columns, using the fewest straight lines connected at their ending points . Ripà and Remirez [6] showed that it is possible to do this for every $n \in \mathbb{N} - \{0, 1, 2\}$, using $2 \cdot n$-2 straight lines only. For any $n > 5$, we can combine a given 8 lines solution for the 5X5 problem and the square spiral frame [10]. In the same paper, they extended the nXn result to a three dimensional space [8] and finally to a generic k-dimensional space (for $k > 3$).

Starting from that outcome, we consider the same problem and rules by [6]. We can apply the "pure" spiral method to a n_1 X n_2 rectangular grid (where $n_1 \leq n_2$). In this way, it is quite simple to find out that the minimum amount of lines we need to connect every point (solving the problem inside the box, connecting points without crossing a line and visiting any dot just once) is given by the **Eq. 1** [9].

$$h = 2 \cdot n - 1 \quad \forall\, n \in \mathbb{N} - \{0, 1\} \qquad \textbf{(1)}$$

2. The n_1Xn_2X…Xn_k problem upper bound

If we try to extend the result by **Eq. 1** to a three dimensional space, where $n_1 \leq n_2 \leq n_3$, we need to modify a little bit the standard strategy described in [6] in order to choose the best "plane by plane" approach we can, even if there can be a few exceptions (such as if $n_3 - n_2 \leq 1$, see Appendix). So, we need to identify the correct starting plane to lay the first straight line. Using basic mathematics, it is quite easy to prove that, in general, the best option is to start from the [n_2 ; n_3] plane.

Hence, under the additional constraints that we must solve the problem inside the box only, connecting points without crossing a line and visiting any dot just once, our strategy is as follows:

Step 1) Take one of the external planes identified by $[n_2 ; n_3]$: here is the plane to lay our first line;

Step 2) Starting from one point on an angle of this grid, draw the first straight line to connect n_3 points, until we have reached the last point in that row;

Step 3) The next line is on the same plane as well. It lays on $[n_2 ; n_3]$, it is orthogonal to the previous one, and it links $n_2 - 1$ points;

Step 4) Repeat the square (rectangular) spiral pattern until we connect every point belonging to this $n_2 \cdot n_3$ set to the others on the same surface;

Step 5) Draw another line which is orthogonal to the $[n_2 ; n_3]$ plane we have considered before, doubling the same scheme (in reverse) with the opposite face of this three dimensional box with the shape of a (n_1 , n_2 , n_3) parallelogram. Repeat the same pattern for any $n_2 \text{X} n_3$ grid, $n_1 - 2$ times more.

The rectangular spiral solution give us also the shortest path we can find to connect every point: the total length of the line segments used to fit all the points is minimal.

N.B.

Just a couple of trivial considerations. Referring to the rectangular spiral pattern applied to a k-dimensional space ($k \geq 2$), we can return to the starting point using exactly one additional line (it works for any number of dimensions at or above 1 we can consider). For any odd value of n_1, we can visit a maximum $\left\lceil \frac{n_{k-1}}{2} \right\rceil - 1$ points twice, simply extending the ending line (if we do not, we will not visit any dot more than once, otherwise we can visit $\left\lceil \frac{n_{k-1}}{2} \right\rceil - 1$ points, at most). Moreover, it is possible to visit up to $n_{k-1} - 2$ points twice if we move the second-last line too (crossing some lines more as well). Finally, considering $k \geq 2$, if we are free to extend the ending line until we are close to the next (already visited) point (i.e., let ε be the distance between the last line and the nearest point and let be the distance between two adjacent points unitary, we have that $0 < \varepsilon < 1$), it is possible to return to the starting point without visiting any point more than once.

The total amount of lines we use to connect every point is always lower or equal to

$$h = 2 \cdot n_1 \cdot n_2 - 1 \qquad \textbf{(2)}$$

In fact, $h = (2 \cdot n_2 - 1) \cdot n_1 + n_1 - 1$.

Nevertheless, $(2 \cdot n_2 - 1) \cdot n_1 + n_1 - 1 = 2 \cdot n_1 \cdot n_2 - n_1 + n_1 - 1 = 2 \cdot n_1 \cdot n_2 - 1 = 2 \cdot n_1 \cdot n_2 - n_2 + n_2 - 1 =$
$= (2 \cdot n_1 - 1) \cdot n_2 + n_2 - 1$ (Q.E.D.).

The "savings", in terms of unused segments, is zero if (and only if)

$$n_1 < 2 \cdot (n_3 - n_2) + 3 \qquad \textbf{(3)}$$

In general, (also if $n_1 \geq 2 \cdot (n_3 - n_2) + 3$), the **Eq. 2** can be rewritten as:

$$h = 2 \cdot n_1 \cdot n_2 - c \qquad \textbf{(4)}$$

Where $c = 1$ if the "savings" is zero, while $c \geq 2$ if not.

As an example, let us consider the following cases:

a) $n_1 = 5$; $n_2 = 6$; $n_3 = 9$.
b) $n_1 = 11$; $n_2 = 12$; $n_3 = 13$.

While in the first hypothesis $c = 1$ (in fact $5 < 2 \cdot (9 - 3) + 3$), $h = 2 \cdot 5 \cdot 6 - 1 = 59$, in case b) we have $c = 13$, $h = 2 \cdot 11 \cdot 12 - 13 = 251$. This is in virtue of the fact that the fifth and the sixth connecting line allow us to "save" one line for every subsequent plane, whereas each plane "met" after the sixth can be solved using two lines less (if compared with the first four we have considered).

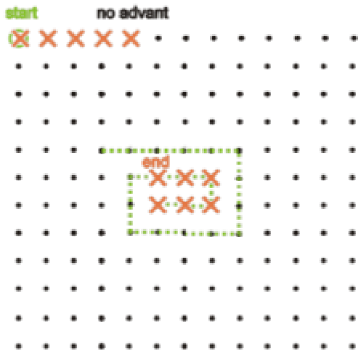

Fig. 2. The rectangular spiral for the case of the example b): $n_1 = 11$, $n_2 = 12$, $n_3 = 13$.

If $n_1 \geq 2 \cdot (n_3 - n_2) + 4$, the (pure) rectangular spiral method, with specific regard to the three dimensional problem, can be summarized as follows:

$h = n_1 - 1 + [2 \cdot (n_3 - n_2) + 2] \cdot (2 \cdot n_2 - 1) + 2 \cdot (2 \cdot n_2 - 2) + [2 \cdot (n_3 - n_2) + 4] \cdot (2 \cdot n_2 - 3) + 4 \cdot (2 \cdot n_2 - 4) + [2 \cdot (n_3 - n_2) + 6] \cdot (2 \cdot n_2 - 5) + 6 \cdot (2 \cdot n_2 - 6) + \ldots + d$

Where d represents the product of the number of line segments used to solve the plane which contains the fewest lines (the last plane we have considered, the plane which cuts roughly halfway our imaginary *box*) and "$n_1 - \{[2 \cdot (n_3 - n_2) + 2] + 2 + [2 \cdot (n_3 - n_2) + 4] + 4 + \cdots\}$".

Thus, we can synthesize the previous formula as

$$h = n_1 - 1 + \sum_{j=0}^{j_{max}} [(2 \cdot n_2 - 2 \cdot j - 1) \cdot (2 \cdot (n_3 - n_2) + 2 \cdot (j+1)) + 2 \cdot (j+1) \cdot (2 \cdot n_2 - 2 \cdot (j+1))] + b$$

Hence

$$h = -\frac{8 \cdot j_{max}^3}{3} + 6 \cdot j_{max}^2 \cdot n_2 - 2 \cdot j_{max}^2 \cdot n_3 - 11 \cdot j_{max}^2 - 4 \cdot j_{max} \cdot n_2^2 + 4 \cdot j_{max} \cdot n_2 \cdot n_3 + 16 \cdot j_{max} \cdot n_2 - 4 \cdot j_{max} \cdot n_3 - \frac{43 \cdot j_{max}}{3} - 4 \cdot n_2^2 + 4 \cdot n_2 \cdot n_3 + 10 \cdot n_2 - 2 \cdot n_3 - 7 + n_1 + b$$

j_{max} represents the <u>maximum</u> value of the upper bound of the summation, let us say $\tilde{j}$, such that
$n_1 \geq \sum_{j=0}^{\tilde{j}} [2 \cdot (n_3 - n_2) + 2 \cdot (j+1) + 2 \cdot (j+1)] \rightarrow n_1 \geq 2 \cdot (\tilde{j}+1) \cdot (n_3 - n_2 + \tilde{j} + 2)$, while

$$b := \begin{cases} [n_1 - 2 \cdot (j_{max}+1) \cdot (n_3 - n_2 + j_{max} + 2)] \cdot (2 \cdot n_2 - 2 \cdot j_{max} - 3) \\ \quad \textbf{\textit{if}} \quad n_1 - 2 \cdot (j_{max}+1) \cdot (n_3 - n_2 + j_{max} + 2) \leq 2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2) \\ \\ [2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2)] \cdot (2 \cdot n_2 - 2 \cdot j_{max} - 3) + \\ \{n_1 - 2 \cdot (j_{max}+1) \cdot (n_3 - n_2 + j_{max} + 2) - [2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2)]\} \cdot (2 \cdot n_2 - 2 \cdot j_{max} - 4) \\ \quad \textbf{\textit{if}} \quad n_1 - 2 \cdot (j_{max}+1) \cdot (n_3 - n_2 + j_{max} + 2) > 2 \cdot (n_3 - n_2) + 2 \cdot (j_{max} + 2) \end{cases}$$

Making some calculations, we have that

$$b := \begin{cases} 4 \cdot j_{max}{}^3 - 8 \cdot j_{max}{}^2 \cdot n_2 + 4 \cdot j_{max}{}^2 \cdot n_3 + 18 \cdot j_{max}{}^2 - 2 \cdot j_{max} \cdot n_1 + 4 \cdot j_{max} \cdot n_2{}^2 - 4 \cdot j_{max} \cdot n_2 \cdot n_3 - \\ 22 \cdot j_{max} \cdot n_2 + 10 \cdot j_{max} \cdot n_3 + 26 \cdot j_{max} + 2 \cdot n_1 \cdot n_2 - 3 \cdot n_1 + 4 \cdot n_2{}^2 - 4 \cdot n_2 \cdot n_3 - 14 \cdot n_2 + 6 \cdot n_3 + 12 \\ \quad \boldsymbol{if} \quad n_1 \leq 2 \cdot (j_{max} + 2) \cdot (j_{max} - n_2 + n_3 + 2) \\ \\ 4 \cdot j_{max}{}^3 - 8 \cdot j_{max}{}^2 \cdot n_2 + 4 \cdot j_{max}{}^2 \cdot n_3 + 20 \cdot j_{max}{}^2 - 2 \cdot j_{max} \cdot n_1 + 4 \cdot j_{max} \cdot n_2{}^2 - 4 \cdot j_{max} \cdot n_2 \cdot n_3 - \\ 24 \cdot j_{max} \cdot n_2 + 12 \cdot j_{max} \cdot n_3 + 34 \cdot j_{max} + 2 \cdot n_1 \cdot n_2 - 4 \cdot n_1 + 4 \cdot n_2{}^2 - 4 \cdot n_2 \cdot n_3 - 18 \cdot n_2 + 10 \cdot n_3 + 20 \\ \quad \boldsymbol{if} \quad n_1 > 2 \cdot (j_{max} + 2) \cdot (j_{max} - n_2 + n_3 + 2) \end{cases}$$

Thus, the general solution is given by:

$$h = \begin{cases} \frac{4 \cdot j_{max}{}^3}{3} - 2 \cdot j_{max}{}^2 \cdot n_2 + 2 \cdot j_{max}{}^2 \cdot n_3 + 7 \cdot j_{max}{}^2 - 2 \cdot j_{max} \cdot n_1 - 6 \cdot j_{max} \cdot n_2 + \\ \quad 6 \cdot j_{max} \cdot n_3 + \frac{35 \cdot j_{max}}{3} + 2 \cdot n_1 \cdot n_2 - 2 \cdot n_1 - 4 \cdot n_2 + 4 \cdot n_3 + 5 \\ \boldsymbol{if} \quad n_1 \leq 2 \cdot (j_{max}{}^2 - j_{max} \cdot n_2 + j_{max} \cdot n_3 + 4 \cdot j_{max} - 2 \cdot n_2 + 2 \cdot n_3 + 4) \\ \\ \frac{4 \cdot j_{max}{}^3}{3} - 2 \cdot j_{max}{}^2 \cdot n_2 + 2 \cdot j_{max}{}^2 \cdot n_3 + 9 \cdot j_{max}{}^2 - 2 \cdot j_{max} \cdot n_1 - 8 \cdot j_{max} \cdot n_2 + \\ \quad 8 \cdot j_{max} \cdot n_3 + \frac{59 \cdot j_{max}}{3} + 2 \cdot n_1 \cdot n_2 - 3 \cdot n_1 - 8 \cdot n_2 + 8 \cdot n_3 + 13 \\ \boldsymbol{if} \quad n_1 > 2 \cdot (j_{max}{}^2 - j_{max} \cdot n_2 + j_{max} \cdot n_3 + 4 \cdot j_{max} - 2 \cdot n_2 + 2 \cdot n_3 + 4) \end{cases} \qquad \textbf{(5)}$$

Where j_{max} is the <u>maximum</u> value $j \in \mathbb{N}_0$ such that $n_1 \geq 2 \cdot [j^2 + (n_3 - n_2 + 3) \cdot j + n_3 - n_2 + 2]$
$\rightarrow j_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{n_3{}^2 + n_2{}^2 - 2 \cdot n_2 \cdot n_3 + 2 \cdot n_3 - 2 \cdot n_2 + 2 \cdot n_1 + 1} + n_2 - n_3 - 3 \right) \right\rfloor$.

The **Eq. 5** can be rewritten more elegantly as

$$h = \begin{cases} \frac{4}{3} \cdot j_{max}{}^3 + [2 \cdot (n_3 - n_2) + 7] \cdot j_{max}{}^2 + \left[6 \cdot (n_3 - n_2) - 2 \cdot n_1 + \frac{35}{3} \right] \cdot j_{max} + 4 \cdot (n_3 - n_2) + 2 \cdot n_1 \cdot (n_2 - 1) + 5 \\ \quad \boldsymbol{if} \quad n_1 \leq 2 \cdot \left[j_{max}{}^2 + (n_3 - n_2 + 4) \cdot j_{max} + 2 \cdot (n_3 - n_2) + 4 \right] \\ \\ \frac{4}{3} \cdot j_{max}{}^3 + [2 \cdot (n_3 - n_2) + 9] \cdot j_{max}{}^2 + \left[8 \cdot (n_3 - n_2) - 2 \cdot n_1 + \frac{59}{3} \right] \cdot j_{max} + 8 \cdot (n_3 - n_2) + n_1 \cdot (2 \cdot n_2 - 3) + 13 \\ \quad \boldsymbol{if} \quad n_1 > 2 \cdot \left[j_{max}{}^2 + (n_3 - n_2 + 4) \cdot j_{max} + 2 \cdot (n_3 - n_2) + 4 \right] \end{cases} \qquad \textbf{(6)}$$

Where $j_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{n_3{}^2 + n_2{}^2 - 2 \cdot n_2 \cdot n_3 + 2 \cdot (n_3 - n_2 + n_1) + 1} + n_2 - n_3 - 3 \right) \right\rfloor$.

N.B.

For obvious reasons, the **Eq. 6** is always applicable, on condition that $n_1 \geq 2 \cdot (n_3 - n_2) + 4$. Otherwise, the solution follows immediately from **Eq. 4**, since c can assume only two distinct values: 1 or 2 (c = 1 if the condition **(3)** is verified, c = 2 if the **(3)** is not satisfied, but the **Eq. 6** cannot be used – therefore, this is the case $n_1 = 2 \cdot (n_3 - n_2) + 3$).

Fig. 3. The rectangular spiral and its development [2] for the cases of (from left to right) $n_3 - n_2 = 0$, $n_3 - n_2 = 1$ and $n_3 - n_2 = 2$.

Therefore, it is possible to extend the aforementioned result we have previously shown to the k-dimensional case: $n_1 \text{X} n_2 \text{X} \ldots \text{X} n_k$. The method to determine an acceptable upper limit for the optimal solution remains the same as the case $n_1 = n_2 = \ldots = n_k$:

$$h = (t+1) \cdot \prod_{j=1}^{k-3} n_j - 1 \qquad \textbf{(7)}$$

Where t, the lowest upper limit available for the $n_{k-2} \text{ X } n_{k-1} \text{ X } n_k$ problem (with the exception of the very particular cases we have introduced at the beginning of the paper [6]), is given by the **Eq. 4** and it is made explicit by the **(2-6)**.

Specifically, we will start considering an external grid defined by $[n_{k-1} \; ; \; n_k]$, we will connect the corresponding $n_k \cdot n_{k-1}$ points using $2 \cdot n_{k-1} - 1$ lines (following the rectangular spiral pattern), from the ending point of that external grid, we will draw the line segment which is orthogonal to any $[n_{k-1} \; ; \; n_k]$ plane (along the n_{k-2} points direction), and so on.

3. The n_1Xn_2X…Xn_k problem bounded from below

In this section we provide a non-trivial lower bound for the k-dimensional $n_1 \text{X} n_2 \text{X} \ldots \text{X} n_k$ points problem. In this way, we can build a range in which certainly will fall all the best possible solutions to the problem we are considering (for any natural number n_i and number of dimensions k). In conclusion, we provide a few characteristic numerical examples in order to appreciate the quality of the result arising from the particular approach we have chosen.

For $k = 3$ ($n_1 \leq n_2 \leq n_3$), let us examine first the structure of the grid: it is not possible to intersect more than $(n_3-1)+(n_2-1)=n_3+n_2-2$ points using two consecutive lines, however, there is one exception (which, for simplicity, we may assume as in the case of the first two lines drawn). In this circumstance, it is possible to fit n_3 points with the first line and n_2-1 points using the second one, just as in the case of the pure rectangular spiral solution that we have already considered.

Let we observe now that, lying (by definition) each segment on a unique plan, it will be necessary to provide n_1-1 lines to connect the various plans that are addressed in succession (of any type): there is no way to avoid spending less than n_1-1 lines to connect (at most) n_1-1 points at a time (under the constraint previously explained above to connect n_3+n_2-1 points with the first two line segments). Each of these lines could be interposed between as many rectilinear line segments capable of connecting n_k-1 points at a time.

Following the same pattern, we notice that the previous result, in the k-dimensions case ($k \geq 3$), does not substantially change.

Let h_l be the number of line segments of our lower bound, for any $k \geq 3$, we have that

$$\prod_{i=1}^{k} n_i :\leq n_k + \sum_{j=1}^{k-2}(n_j-1)^2 + (n_k-1)\cdot\sum_{j=1}^{k-2}(n_j-1) + \left[h_l - 2\cdot\sum_{j=1}^{k-2}(n_j-1) - 1\right]\cdot\left\lfloor\frac{n_k+n_{k-1}}{2} - 1\right\rfloor \qquad (\mathbf{8})$$

Taking account of the fact that, $\forall\, n_k, n_{k-1}$, $\left\lfloor\frac{n_k+n_{k-1}}{2} - 1\right\rfloor \leq \left\lfloor\frac{n_k+n_{k-1}-1}{2}\right\rfloor$, doing some basic calculations, we get the following result:

$$\begin{cases} h_l \geq \left\lceil \dfrac{2}{n_k+n_{k-1}-2}\cdot\left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j^{\,2} + (3-n_k)\cdot\sum_{j=1}^{k-2} n_j + n_k\cdot(k-3) - 2\cdot k + 4 + (n_k+n_{k-1}-2)\cdot\sum_{j=1}^{k-2}(n_j-1)\right]\right\rceil + 1 \\ \qquad\qquad \boldsymbol{if}\quad \dfrac{n_k+n_{k-1}}{2} \in \mathbb{N}\backslash\{0,1\} \\ h_l \geq \left\lceil \dfrac{2}{n_k+n_{k-1}-1}\cdot\left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j^{\,2} + (3-n_k)\cdot\sum_{j=1}^{k-2} n_j + n_k\cdot(k-3) - 2\cdot k + 4 + (n_k+n_{k-1}-1)\cdot\sum_{j=1}^{k-2}(n_j-1)\right]\right\rceil + 1 \\ \qquad\qquad \boldsymbol{if}\quad \dfrac{n_k+n_{k-1}+1}{2} \in \mathbb{N}\backslash\{0,1\} \end{cases}$$

Hence

$$h_l \geq \begin{cases} \left\lceil \dfrac{2}{n_k+n_{k-1}-2}\cdot\left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j^{\,2} + \sum_{j=1}^{k-2} n_j - n_k + n_{k-1}\cdot\left(\sum_{j=1}^{k-2} n_j - k + 2\right)\right]\right\rceil + 1 \\ \qquad\qquad \boldsymbol{if}\quad \dfrac{n_k+n_{k-1}}{2} \in \mathbb{N}\backslash\{0,1\} \\ \left\lceil \dfrac{2}{n_k+n_{k-1}-1}\cdot\left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} n_j^{\,2} + 2\cdot\sum_{j=1}^{k-2} n_j - n_k + n_{k-1}\cdot\left(\sum_{j=1}^{k-2} n_j - k + 2\right) - k + 2\right]\right\rceil + 1 \\ \qquad\qquad \boldsymbol{if}\quad \dfrac{n_k+n_{k-1}+1}{2} \in \mathbb{N}\backslash\{0,1\} \end{cases} \qquad (\mathbf{9})$$

Notice now how we can improve the result by the **(9)** whereas the linking lines between the various plans cannot actually intersect n_i-1 points each: to connect all the points of every plane belonging to the dimension/s distinguished by the fewest points aligned (the values of the n_i characterized by the lowest subscript) it is possible to connect n_i-1 points with the first line segment, n_i-2 using the second line segment, n_i-3 points with the next one, and so on.

Therefore, with reference to the three-dimensional case, these n_1-1 linking lines intersect $\sum_{j=1}^{n_1-1}(n_1-j) = \frac{n_1\cdot(n_1-1)}{2}$ new (unvisited) points. As noted above, we can assume that, at most, each one of them will precede and follow as many line segments that intersect n_k-1 points.

Thus

$$\prod_{i=1}^{k} n_i :\leq n_k + \frac{1}{2}\cdot\sum_{j=1}^{k-2} n_j\cdot(n_j-1) + (n_k-1)\cdot\sum_{j=1}^{k-2}(n_j-1) + \left[h_l - 2\cdot\sum_{j=1}^{k-2}(n_j-1) - 1\right]\cdot\left\lfloor\frac{n_k+n_{k-1}}{2} - 1\right\rfloor \qquad (\mathbf{10})$$

Hence

$$h_l \geq \begin{cases} \left\lceil \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) + k - 2 \right] \right\rceil + 1 \\ \qquad \boldsymbol{if} \quad \frac{n_k + n_{k-1}}{2} \in \mathbb{N} \backslash \{0,1\} \\ \left\lceil \frac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 + \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) \right] \right\rceil + 1 \\ \qquad \boldsymbol{if} \quad \frac{n_k + n_{k-1} + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases} \qquad (\mathbf{11})$$

In detail (looking at the **(11)**), if k =3, it follows that

$$h_l \geq \begin{cases} \left\lceil \frac{2 \cdot n_1 \cdot n_2 \cdot n_3 - {n_1}^2 + 2 \cdot n_1 \cdot n_2 - n_1 - 2 \cdot n_2 - 2 \cdot n_3 + 2}{n_3 + n_2 - 2} \right\rceil + 1 \\ \qquad \boldsymbol{if} \quad \frac{n_3 + n_2}{2} \in \mathbb{N} \backslash \{0,1\} \\ \left\lceil \frac{2 \cdot n_1 \cdot n_2 \cdot n_3 - {n_1}^2 + 2 \cdot n_1 \cdot n_2 + n_1 - 2 \cdot n_2 - 2 \cdot n_3}{n_3 + n_2 - 1} \right\rceil + 1 \\ \qquad \boldsymbol{if} \quad \frac{n_3 + n_2 + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases} \qquad (\mathbf{12})$$

Now, let us consider that, for every $n_k \geq n_{k-1} \geq \ldots \geq n_2 \geq n_1 \geq 2$ ($\forall\, n_i \in \mathbb{N} \backslash \{0,1\}$),

$\frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) + k - 2 \right] \geq \frac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 + \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) \right]$.

Thus, considering the fact that we can arbitrary change the value of n_k (i.e., we can take $\widetilde{n_k} \coloneqq n_k + 1$ if we like) without vary the number of line segments we need to connect every point, we can assume, without loss of generality, that

$$h_l \geq \left\lceil \frac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 - \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 3 \right) + k - 4 \right] \right\rceil - 1 \qquad (\mathbf{13})$$

for any $[n_k, n_{k-1}, \ldots, n_2, n_1]$.

Consequently, if $k = 3$, $h_l \geq \left\lceil \frac{n_1 \cdot (2 \cdot n_2 \cdot n_3 - n_1 + 2 \cdot n_2 - 1) - 2}{n_3 + n_2 - 2} \right\rceil - 1 \qquad (\mathbf{14})$

4. Conclusion

Given $k = 3$, by combining **Eq. 14** with the **(1)-(6)**, we get the intervals in which the best possible solutions of the problem certainly fall.

How wide this range is (and therefore how interesting this outcome may be considered) also depends on the particular values of n_1, n_2 and n_3.

Example 1: $n_1 = 10, n_2 = 13, n_3 = 15$.

$$155 \leq h \leq 253$$

Example 2: $n_1 = 10, n_2 = 21, n_3 = 174$.

$$380 \leq h \leq 419$$

If $k > 3$, the interval is given by

$$\left\lceil \frac{2}{n_k + n_{k-1} - 2} \cdot \left\lceil \prod_{i=1}^{k} n_i - \frac{1}{2} \cdot \left(\sum_{j=1}^{k-2} {n_j}^2 + \sum_{j=1}^{k-2} n_j \right) + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 3 \right) + k - 4 \right\rceil \right\rceil - 1 \leq h \leq (t+1) \cdot \prod_{j=1}^{k-3} n_j - 1$$

Where t, the lowest upper limit for the $n_{k-2} \mathrm{X} n_{k-1} \mathrm{X} n_k$ points problem, is the result obtained from the **(1)-(6)** [6].

In this case, how great is the interval depends also on the particular value of k (in general, the lager the k, the wider is the interval).

Example 3: $k = 4$; $n_1 = 10, n_2 = 16, n_3 = 18, n_4 = 48$ (thus $t = 575$).

$$4328 \leq h \leq 5759$$

If I had to gamble, setting $k := 3$, I would bet on any betting odds higher than $1+10^{-80}$: 1(there are roughly 10^{80} atoms in the visible universe) that "h_{best}", the number of straight line segments associated with the best possible solution, is significantly closer to the upper bound I defined than can be small compared to its counterpart (mathematically, I would be willing to bet on the fact that, for the vast majority of the possible combinations $[n_1, n_2, n_3]$, $\frac{h_u - h_{best}}{h_{best} - h_l} < 1$).

It is interesting to note that, for some particular combinations, the upper bound and the lower bound coincide, thus allowing us to obtain a complete and definitive resolution of the given problem.

E.g., for $k = 3$; $n_1 = n_2 = 3, n_3 = 61$, it follows that $h_l = h_u = h_{best} = 17$. Ditto if $k = 3$; $n_1 = 3, n_2 = 4, n_3 = 57$. In fact, $h_l = h_u = h_{best} = 23$. While, if $k = 4$; ; $n_1 = n_2 = n_3 = 2, n_4 = 46$, $h_l = h_u = h_{best} = 15$.

5. Appendix

If we do not take into account all the original constraints (solving the problem "inside the box" only, no intersections between lines, and so on) we could improve our "plane by plane" upper bound. For example, we could use the basic pattern below (**Fig. 4**), for any $n \geq 4$. This kind of solutions can be applied to the n X n X … X n points problem and to the n_1 X n_2 X … X n_k points one as well (e.g., n_k - n_{k-1} = 1→ **Fig. 5**):

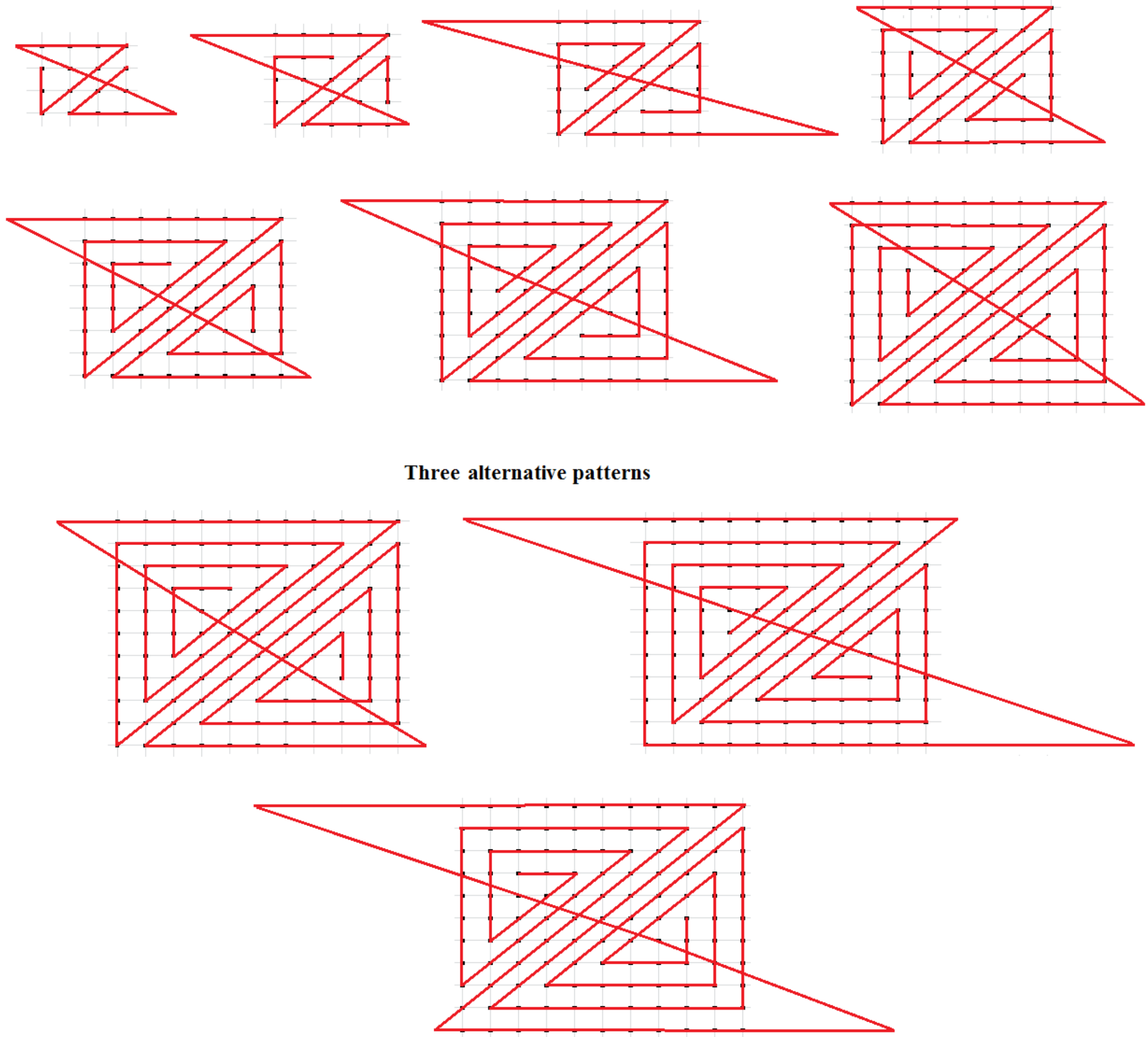

Fig. 4. The "double spiral" pattern for $n_k = n_{k-1}$ ($2 \cdot n_{k-1}$-2 lines).

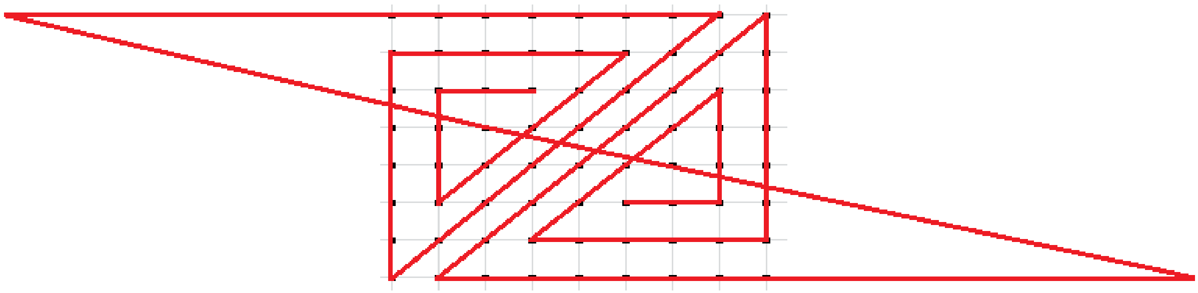

Fig. 5. The “double spiral” pattern if $n_k - n_{k-1} = 1$ ($2 \cdot n_{k-1}$-1 lines).

Looking at the pattern of **Fig. 5**, we can easily discover that we can use it to reduce the 3D upper bound by the rectangular spiral: e.g., for $n_1 = n_2 = 22$, $n_3 = 23$ it follows that $h_u = 902$, which is far better than 917, the rectangular spiral one.

Therefore, if $n_1 = n_2 = n_3$, the best “thinking outside the box” upper bounds are as follows:

Table 1: *n*X*n*X*n* points puzzle upper bounds following the “double spiral pattern” by **Fig. 4**.

n	Best Upper Bound Currently Discovered	*n*	Best Upper Bound Currently Discovered	*n*	Best Upper Bound Currently Discovered
1	/	18	**587**	35	**2258**
2	**7**	19	**655**	36	**2391**
3	**14**	20	**726**	37	**2528**
4	**26**	21	**801**	38	**2669**
5	**42**	22	**880**	39	**2814**
6	**62**	23	**963**	40	**2963**
7	**85**	24	**1050**	41	**3115**
8	**112**	25	**1141**	42	**3270**
9	**143**	26	**1236**	43	**3429**
10	**178**	27	**1335**	44	**3592**
11	**216**	28	**1438**	45	**3759**
12	**257**	29	**1544**	46	**3930**
13	**302**	30	**1653**	47	**4105**
14	**351**	31	**1766**	48	**4284**
15	**404**	32	**1883**	49	**4467**
16	**461**	33	**2004**	50	**4654**
17	**522**	34	**2129**	51	**4845**

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