## DANZER MATRICES

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The purpose of this note is to give a definition, as briefly as possible, of Danzer matrices ${ }^{1}$ (formerly called "unit-primitive" matrices) and the matrix basis formed from them. It is a plan for the future to expand the text presented here to include a spectral theory, based in part on suggestions made by Prof. Lorenzo Sadun [3], along with inclusion of the direct application of Danzer matrices to the theory of tiles, as well as their association with some sequences found in The Online Encyclopedia of Integer Sequences (OEIS) [4].

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. Let $n \in \mathbb{N}$, $n>3$, and $q_{n}=\left\lfloor\frac{n}{2}\right\rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function. Let $R$ be a ring with multiplicative identity $1_{R}$ and additive identity $0_{R}$. Let $x \in R$, and define the recurrence

$$
\begin{aligned}
& S_{0}(x)=1_{R} \\
& S_{1}(x)=x \\
& S_{r}(x)=x S_{r-1}(x)-S_{r-2}(x) \quad(r=2,3, \ldots)
\end{aligned}
$$

We use the same notation as that for the "Chebyshev $S$-polynomials" defined by Wolfdieter Lang in [2], since here only the initial conditions differ.

For any pair of indices $u, v \in \mathbb{N} \cup\{0\}$, the product $S_{u}(x) S_{v}(x)$ has representation as an integral linear combination of $S$-polynomials of the form

$$
S_{u}(x) S_{v}(x)=\sum_{l=0}^{\min (u, v)} S_{|u-v|+2 l}(x)
$$

where $|\cdot|$ means absolute value. The $S$-polynomials pairwise commute, so the order in which the indices $u, v$ are taken in the product is irrelevant.

Let $T_{n}$ be the $q_{n} \times q_{n}$ tridiagonal matrix

$$
T_{n}=\left\{\begin{array}{l}
{\left[\begin{array}{ccccc}
0 & 1 & & & \\
1 & 0 & 1 & & \\
& \ddots & \ddots & \ddots & \\
& & 1 & 0 & 1 \\
& & & 1 & 1
\end{array}\right], \quad \text { if } n \text { is odd }} \\
{\left[\begin{array}{lllll}
0 & 1 & & & \\
1 & 0 & 1 & & \\
& \ddots & \ddots & \ddots & \\
& & 1 & 0 & 1 \\
& & & 2 & 0
\end{array}\right], \quad \text { if } n \text { is even }}
\end{array}\right.
$$

[^0]with all remaining entries set to zero. Let $I_{q_{n}}$ and $0_{q_{n}}$ denote the $q_{n} \times q_{n}$ identity and zero matrices, respectively. Let
$$
\mathcal{M}_{n}=\left\{a_{0} I_{q_{n}}+a_{1} T_{n}+\cdots+a_{k} T_{n}^{k}: a_{j} \in \mathbb{Z}, k \in \mathbb{N} \cup\{0\}, j=0, \ldots, k\right\}
$$
be the set of all polynomials in $T_{n}$ with coefficients in $\mathbb{Z}$. Any element of the ring $\mathcal{M}_{n}$ has equivalent representation in the matrix basis $\left\{I_{q_{n}}, T_{n}, \ldots, T_{n}^{q_{n}-1}\right\}$. However, we instead consider a "nicer" basis defined as follows.

Under certain conditions the resulting infinite sequence $\left\{S_{r}(x)\right\}_{r=0}^{\infty}$ can be periodic. Indeed this is true for the $q_{n} \times q_{n}$ matrix $T_{n}$ for which the sequence $\left\{S_{r}\left(T_{n}\right)\right\}_{r=0}^{\infty}$ turns out to be periodic ${ }^{2}$ with period $2 n$, for all $n$. In this case we have for the index $r$ of $S_{r}\left(T_{n}\right)$ that $r \equiv h(\bmod 2 n)$, with $r=2 n s+h$ for some $s \in \mathbb{N} \cup\{0\}$ and $h \in\{0,1, \ldots, 2 n-1\}$.

It is therefore convenient to first reduce modulo $2 n$ the index $r$ of $S_{r}\left(T_{n}\right)$, putting the polynomial equal to one residing somewhere in the "fundamental block" comprising the subsequence

$$
\mathcal{S}=\left\{S_{0}\left(T_{n}\right), S_{1}\left(T_{n}\right), \ldots, S_{2 n-1}\left(T_{n}\right)\right\}
$$

of the sequence $\left\{S_{r}\left(T_{n}\right)\right\}_{r=0}^{\infty}$. The block $\mathcal{S}$ can be viewed as a multiset with the following structure, in which we put $S_{k}=S_{k}\left(T_{n}\right)$ for brevity. If $n$ is odd, then
$2 n$ terms
$\mathcal{S}=\overbrace{\left\{S_{0}, \ldots, S_{q_{n}-1}, S_{q_{n}-1}, \ldots, S_{0}, 0_{q_{n}},-S_{0}, \ldots,-S_{q_{n}-1},-S_{q_{n}-1}, \ldots,-S_{0}, 0_{q_{n}}\right\}} ;$
and if $n$ is even, then

$$
\mathcal{S}=\overbrace{\left\{S_{0}, \ldots, S_{q_{n}-1}, \ldots, S_{0}, 0_{q_{n}},-S_{0}, \ldots,-S_{q_{n}-1}, \ldots,-S_{0}, 0_{q_{n}}\right\}}^{2 n \text { terms }},
$$

the latter block in which the two terms $\pm S_{q_{n}-1}\left(T_{n}\right)$ are not repeated as in the former block. For example, for $n=7$ (so $q_{7}=3$ ), still using our abbreviated notation,

$$
\mathcal{S}=\left\{S_{0}, S_{1}, S_{2}, S_{2}, S_{1}, S_{0}, 0_{3},-S_{0},-S_{1},-S_{2},-S_{2},-S_{1},-S_{0}, 0_{3}\right\}
$$

with the block length being equal to $2 n=2 \times 7=14$. Similarly, for $n=6\left(q_{6}=3\right)$, we get the block

$$
\mathcal{S}=\left\{S_{0}, S_{1}, S_{2}, S_{1}, S_{0}, 0_{3},-S_{0},-S_{1},-S_{2},-S_{1},-S_{0}, 0_{3}\right\}
$$

Note in particular that $S_{r}\left(T_{n}\right)=0_{q_{n}}$ if and only if $r+1 \equiv 0(\bmod n)$. More generally, for any reduced index $r \equiv h \in\{0, \ldots, 2 n-1\}$, precisely one of the following identities holds:

$$
S_{r}\left(T_{n}\right)= \begin{cases}S_{h}\left(T_{n}\right), & \text { if } 0 \leq h<q_{n} \\ S_{n-h-2}\left(T_{n}\right), & \text { if } q_{n} \leq h<n-1 \\ -S_{h-n}\left(T_{n}\right), & \text { if } n \leq h<n+q_{n} \\ -S_{2 n-h-2}\left(T_{n}\right), & \text { if } n+q_{n} \leq h<2 n-1 \\ 0_{q_{n}}, & \text { if } h=n-1 \text { or } 2 n-1\end{cases}
$$

It follows, for any index $r \in \mathbb{N} \cup\{0\}$, that $S_{r}\left(T_{n}\right)$ can be resolved to precisly one element of the set $\left\{0_{q_{n}}, \pm S_{0}\left(T_{n}\right), \ldots, \pm S_{q_{n}-1}\left(T_{n}\right)\right\}$. We take the positive $S$ 's from

[^1]this set for our matrices $D_{N}=S_{N}\left(T_{n}\right)$, for $N=0, \ldots, q_{n}-1$, which I call Danzer matrices, and the totality of the $D_{N}$ as the ordered set $\mathcal{D}_{n}=\left\{D_{0}, \ldots, D_{q_{n}-1}\right\}$, which I call the Danzer basis.

Since any matrix $M \in \mathcal{M}_{n}$ can be represented as an integral linear combination of $S$-polynomials (in $T_{n}$ ), it follows that $M$ has equivalent representation in the Danzer basis of the form

$$
M=b_{0} D_{0}+\cdots+b_{q_{n}-1} D_{q_{n}-1},
$$

for some $b_{0}, \ldots, b_{q_{n}-1} \in \mathbb{Z}$. Moreover, if $M=\left(m_{i, j}\right), i, j=0, \ldots, q_{n}-1$, then $m_{0, j}=b_{j}$, for all $j$, that is, the $b_{j}$ are just the first row entries of $M$. For example, for the matrix $M=\left(\begin{array}{ll}1 & 1 \\ 1 & 2\end{array}\right)$ associated with the Penrose substitution rule [1] for the rhombus tilings for $n=5$, we have (since $q_{5}=2$ ) that

$$
M=\left(m_{i, j}\right)=\left(\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right)=D_{1}^{2}=m_{0,0} D_{0}+m_{0,1} D_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right) .
$$

Finally, for $n=3,4, \ldots$, and for the purpose of illustration, we list the Danzer basis corresponding to each the first few $n$ :

$$
\begin{aligned}
& \mathcal{D}_{3}=\{[1]\} \\
& \mathcal{D}_{4}=\left\{\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right],\left[\begin{array}{ll}
0 & 1 \\
2 & 0
\end{array}\right]\right\} \\
& \mathcal{D}_{5}=\left\{\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right],\left[\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right]\right\} \\
& \mathcal{D}_{6}=\left\{\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right],\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0
\end{array}\right],\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 2 & 0 \\
2 & 0 & 1
\end{array}\right]\right\} \\
& \mathcal{D}_{7}=\left\{\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right],\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{array}\right],\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}\right]\right\} \\
& \mathcal{D}_{8}=\left\{\left[\begin{array}{llll}
1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right],\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0
\end{array}\right],\left[\begin{array}{llll}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 2 & 0 \\
0 & 2 & 0 & 1
\end{array}\right],\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 \\
0 & 2 & 0 & 1 \\
2 & 0 & 2 & 0
\end{array}\right]\right\} \\
& \mathcal{D}_{9}=\left\{\left[\begin{array}{llll}
1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right],\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{array}\right],\left[\begin{array}{llll}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{array}\right],\left[\begin{array}{lllll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right]\right\}
\end{aligned}
$$

in which, for the sake of completeness, we have also defined the set consisting of the singleton $1 \times 1$ scalar "identity matrix" to be the Danzer basis corresponding to $n=3$.

## References

[1] D. Frettlöh, http://tilings.math.uni-bielefeld.de/substitution_rules/penrose_rhomb
[2] W. Lang, http://www-itp.particle.uni-karlsruhe.de/~wl/EISpub/A049310appl.pdf.
[3] L. Sadun, Personal communications, 2010, 2011.
[4] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, http://oeis.org/.
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[^0]:    ${ }^{1}$ Named after the late mathematician Ludwig Danzer whose work in tiling theory inspired me.

[^1]:    ${ }^{2}$ The reason is related to a certain algebraic property of the spectrum of $T_{n}$.

