On The Sharpness of The Lower Bound

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Theorem . If $A220470(n) = n^2 + n$, then $n+1$ is a power of a prime. Furthermore, any G_n will have an elementary abelian normal subgroup K with $|K| = n+1$, and all nonidentity elements of K will be conjugate in G_n.

Proof. (From here on, we fix an arbitrary G_n and call it G for convenience.) First of all, note that $A220470(1) = 1 < 1^2 + 1$, so the condition implies $n > 1$. Then, since $n > 1$, $|G| = n^2 + n < 2n^2$ so G cannot have two irreducible characters of degree n. By assumption, G has one irreducible character χ of degree n, so χ is unique and well-defined. Since χ is unique, it is invariant under the action of $\mathrm{Gal}(\mathbb{Q}(e^{\frac{2\pi i}{n^2+n}})/\mathbb{Q})$ on character values. Therefore, for all $g \in G$, $\chi(g) \in \mathbb{Q}$. Since character values are algebraic integers, for all $g \in G$, $\chi(g) \in \mathbb{Z}$. The orthogonality relations tell us:

$$1 =< \chi, \chi > \tag{1}$$

$$= \frac{1}{n^2+n} \sum_{g \in G} \chi(g)^2 \qquad \text{(since } \chi \text{ is real-valued)} \tag{2}$$

$$n^2 + n = \sum_{g \in G} \chi(g)^2 \tag{3}$$

$$n = \sum_{g \in G \setminus \{1\}} \chi(g)^2 \tag{4}$$

Since $\chi(1) = n > 1$, χ is a nontrivial character. Then orthogonality relations also tell us (using χ_1 to denote the trivial character):

$$0 =< \chi, \chi_1 > \tag{5}$$

$$0 = |G| < \chi, \chi_1 > \tag{6}$$

$$= \sum_{g \in G} \chi(g) \tag{7}$$

$$-n = \sum_{g \in G \setminus \{1\}} \chi(g) \tag{8}$$

Then adding (4) and (8) gives us:

$$0 = \sum_{g \in G \setminus \{1\}} \chi(g)^2 + \chi(g) \tag{9}$$

Since $\chi(g) \in \mathbb{Z}$ for all $g \in G$,

$$\chi(g)^2 + \chi(g) \geq 0 \tag{10}$$

Then (9) and (10) together imply that

$$\chi(g)^2 + \chi(g) = 0 \tag{11}$$

for all $g \in G \setminus \{1\}$. This means that if $g \neq 1$, $\chi(g) = -1$ or $\chi(g) = 0$.
Let $S = \{g \in G \mid \chi(g) = -1\}$ and $C = \{g \in G \mid \chi(g) = 0\}$.
Now consider the regular character of G:

$$\chi_{reg}(g) = |G|\delta_{1,g} \tag{12}$$

Another bit of notation is in order. Let the number of conjugacy classes of G be c, so that the irreducible characters of G are $\chi_1, \ldots, \chi_c = \chi$. Then the decomposition of χ_{reg} into irreducibles is:

$$\chi_{reg} = \sum_{i=1}^{c} \chi_i(1)\chi_i \tag{13}$$

Therefore

$$\chi_{reg} - n\chi = \sum_{i=1}^{c-1} \chi_i(1)\chi_i \tag{14}$$

is a character of G.
$\chi_{reg}(1) - n\chi(1) = n^2 + n - n^2 = n$. If $g \in S$, then $\chi_{reg}(g) - n\chi(g) = 0 - n(-1) = n$. On the other hand, if $g \in C$, $\chi_{reg}(g) - n\chi(g) = 0 - n \cdot 0 = 0$, so $\ker(\chi_{reg} - n\chi) = \{1\} \cup S$. This is the normal subgroup K whose existence was claimed.
As to the value of $|K|$, note that $G \setminus \{1\} = S \cup C$. Since χ vanishes on C,

$$|S| = \frac{\sum_{g \in S} \chi(g)}{-1} = \frac{\sum_{g \in G \setminus \{1\}} \chi(g)}{-1} = \frac{-n}{-1} = n \tag{15}$$

Then $|K| = 1 + |S| = n + 1$, as claimed.
Since $\ker(\chi_{reg} - n\chi) = K$, every element of K acts trivially on each irreducible component of $\chi_{reg} - n\chi$. Therefore, for all i with $1 \leq i \leq c - 1$, $K \subseteq \ker(\chi_i)$. Then χ_i is constant on S for all i, with $1 \leq i \leq c$. Since all irreducible characters of G are constant on S, all elements of S are conjugate in G. Then K is a normal subgroup of G in which all nonidentity elements are conjugate, so $\mathrm{Aut}(K)$ acts transitively on S. This means that $|K| = n + 1$ is a power of a prime p, and K is an elementary abelian p-group. $\square$