The smallest group with an irreducible representation of dimension 19

David Harden

The purpose of this note is to prove that if G is a finite group with an irreducible 19-dimensional complex representation, then $|G| \geq 3420$. By the discussion on the OEIS page for sequence A220470, it suffices to prove that G has no irreducible 19-dimensional representation when $19 \mid |G|$ and $380 \leq |G| \leq 3401$.

Since $3420 < 19^3$, the Sylow 19-subgroups of any such G are abelian. The same comparison also implies that if the Sylow 19-subgroups of G have order 19^2, their index is less than 19 and therefore G has no irreducible 19-dimensional representation. Therefore we assume that $19 \parallel |G|$ and write $|G| = 19 \cdot n_{19}(G) \cdot |N_G(P)/P|$, where $P \in \mathrm{Syl}_{19}(G)$ and $\mathrm{n}_{19}(G)$ denotes the number of Sylow 19-subgroups of G. $\mathrm{n}_{19}(G) \leq \frac{|G|}{19} \leq 179$ and $\mathrm{n}_{19}(G) \equiv 1 \pmod{19}$, so $\mathrm{n}_{19}(G) = 1, 20, 39, 58, 77, 96, 115, 134, 153,$ or 172.

If $96 \leq \mathrm{n}_{19}(G) \leq 172$, then $\frac{|G|}{19 \cdot \mathrm{n}_{19}(G)} \leq \frac{3401}{19 \cdot 96} < 2$ so the Sylow 19-subgroups of G are self-normalizing. Since they are abelian (in fact, cyclic), this means any Sylow 19-subgroup of G is in the center of its normalizer. Then Burnside's p-Complement Theorem says that G has a normal 19-complement N. If $d \mid |N|$ and $d \equiv 1 \pmod{19}$, then also $\frac{|N|}{d} \mid |N|$ and $\frac{|N|}{d} \equiv 1 \pmod{19}$. Since $\min(d, \frac{|N|}{d}) \leq \sqrt{|N|} < 14$, $\min(d, \frac{|N|}{d}) \equiv 1 \pmod{19}$ implies $\min(d, \frac{|N|}{d}) = 1$, so that $d = 1$ or $|N|$. Since N doesn't centralize a Sylow 19-subgroup of G, it follows that N must have a fixed-point-free automorphism of order 19. But no group of order $96 = 2^5 \cdot 3$, $115 = 5 \cdot 23$, $134 = 2 \cdot 67$, $153 = 3^2 \cdot 17$, or $172 = 2^2 \cdot 43$ has a fixed-point-free automorphism of order 19. This contradiction means we henceforth assume that $\mathrm{n}_{19}(G) \leq 77$.

If $\mathrm{n}_{19}(G) = 77$, then $|N_G(P)/P| \leq \frac{3401}{19 \cdot 77} < 3$, so $|G| = 19 \cdot 77 = 1463$ or $19 \cdot 77 \cdot 2 = 2926$. If $|G| = 2926$, then Isaacs' Theorem 1.35 implies G has a subgroup of index 2. Therefore, in either case, G has a subgroup of order 1463, but any group of order 1463 has a normal Sylow 7-subgroup, the quotient by which is a cyclic group of order $209 = 11 \cdot 19$. Therefore any group of order 1463 is cyclic, giving us a contradiction.

If $\mathrm{n}_{19}(G) = 58$, then $|N_G(P)/P| \leq \frac{3401}{19 \cdot 58} < 4$, so $|G| = 1102, 2204,$ or 3306. In any of these cases, G has a normal Sylow 29-subgroup Q, with $|G/Q| = 38, 76,$ or 114. The normal Sylow 19-subgroup of G/Q lifts to a normal cyclic subgroup of order 551 in G, so that G has a unique Sylow 19-subgroup, contradicting our assumption.

If $\mathrm{n}_{19}(G) = 39$, then $|N_G(P)/P| \leq \frac{3401}{19 \cdot 39} < 5$, so $|G| = 741, 1482, 2223,$ or 2964. In any of these cases, G has a normal Sylow 13-subgroup Q, with $|G/Q| = 57, 114, 171,$ or 228. Then the normal Sylow 19-subgroup of G/Q lifts to a normal cyclic subgroup of order 247 in G, so that G has a unique Sylow 19-subgroup, contradicting our assumption.

If $\mathrm{n}_{19}(G) = 20$, then $|N_G(P)/P| \leq \frac{3401}{19 \cdot 20} < 9$. Let G act by conjugation on its Sylow

19-subgroups. This action is transitive. In fact, it is doubly transitive, since any Sylow 19-subgroup acts without fixed points, and therefore via a 19-cycle, on the others. Any group that acts doubly transitively on 20 points has an irreducible 19-dimensional representation, whose character is obtainable by subtracting the trivial character from the permutation character of the 20-point action. So replacing G by the group of permutations by which G acts on $\mathrm{Syl}_{19}(G)$ if necessary, we may assume G acts faithfully on $\mathrm{Syl}_{19}(G)$. Then we regard G as a permutation group on 20 points. Since a point stabilizer is a Sylow 19-subgroup normalizer and it acts faithfully on 19 points, it is isomorphic to a subgroup of $AGL_1(19)$. Therefore $|N_G(P)/P| \mid 18$, so $|N_G(P)/P| = 1, 2, 3$, or 6. G acts doubly transitively, and therefore primitively, on 20 points. Since 20 is not a power of a prime, G is unsolvable.
If $|G| = 760$ or 2280, then a point stabilizer is isomorphic, respectively, to a subgroup of order 38 or 114 in $AGL_1(19)$. An element of order 2 in a point stabilizer has cycle shape 2^9 so it is an odd permutation. Therefore, if $|G| = 760$ or 2280, G has a subgroup of index 2. Any group of order 380 is solvable, giving a contradiction if $|G| = 380$ or 760. If $|G| = 1140$ or 2280, then G has a subgroup of order 1140. The final contradiction when $\mathrm{n}_{19}(G) > 1$ is therefore obtained by considering $|G| = 1140$.
If $|G| = 1140$, let Q be a Sylow 5-subgroup of G. Since any 3-point stabilizer in G is trivial, the elements of order 5 in Q act as derangements on 20 points. Write $1140 = 5 \cdot \mathrm{n}_5(G) \cdot |N_G(Q)/Q|$, where $\mathrm{n}_5(G)$ denotes the number of Sylow 5-subgroups of G. $|N_G(Q)/Q| \mid 228$, but any element of order 19 in G that normalizes a Sylow 5-subgroup of G centralizes it. But the subgroups of order 19 in S_{20} are self-centralizing, so no elements of order 19 are in $N_G(Q)$. Therefore $|N_G(Q)/Q| \mid 12$. But since $|N_G(Q)/Q| \equiv 3 \pmod 5$, we must have $|N_G(Q)/Q| = 3$. Then $|N_G(Q)| = 15$ so $N_G(Q)$ is cyclic, but all elements of order 5 in G have cycle shape 5^4 and all elements of order 3 in G have cycle shape 3^6. Since, on 20 points, no permutation of cycle shape 5^4 can commute with a permutation of cycle shape 3^6, this is a contradiction. Henceforth we assume that $\mathrm{n}_{19}(G) = 1$, which means P is a normal subgroup of G.

For most orders $|G|$ with $380 \leq |G| \leq 3401$ and $19 \parallel |G|$, it is straightforward to use Sylow theorems and Isaacs' Theorem 1.35 to prove that if G has a normal Sylow 19-subgroup, then G has an abelian subgroup A with $[G : A] < 19$. This implies G has no irreducible 19-dimensional representation. Five difficult cases that arise, however, are given by $|G| = 2052$, $|G| = 2280$, $|G| = 2736$, $|G| = 3040$, and $|G| = 3192$. These are the cases that are examined in detail in this note.

Theorem I. f G is a finite group with an irreducible 19-dimensional representation and $|G| \in \{2052, 2280, 2736, 3040, 3192\}$, then G has no subgroup of index 2.

Proof. Suppose that G contains a subgroup K of index 2. Note that since $|K| = \frac{|G|}{2} \leq 1596 < 2052$, any group of order $|K|$ having a normal Sylow 19-subgroup has an abelian subgroup of index under 19.
Let χ be the character of an irreducible 19-dimensional representation of G. The preceding observation means that the restriction $\chi \downarrow K$ is reducible.
If χ is non-vanishing somewhere outside of K, then, by Theorem 20.12 of James and Liebeck,

the restriction $\chi \downarrow K$ is irreducible, for a contradiction.
If $\chi(g) = 0$ for all $g \in G \setminus K$, then, again by Theorem 20.12 of James and Liebeck, $\chi \downarrow K = \alpha + \beta$, where $\alpha(1) = \beta(1)$. This forces $\alpha(1) = \frac{19}{2}$, which is not an algebraic integer. This contradiction proves that G has no subgroup of index 2. □

Since the Sylow 19-subgroup P is a normal abelian subgroup of G, we get a homomorphism $\phi : G/P \to \mathrm{Aut}(P) \cong \mathbb{F}_{19}^{\times}$. If $\phi(G/P)$ has a subgroup of index 2, that lifts to a subgroup of index 2 in G/P, which lifts to a subgroup of index 2 in G. Therefore $\phi(G/P) \subseteq \{1, 4, 5, 6, 7, 9, 11, 16, 17\}$, which is the Sylow 3-subgroup of $\mathbb{F}_{19}^{\times}$. This means P is centralized by any $3'$-element of G.

1 Order 2052

Theorem I. f $|G| = 2052$ and G has a normal Sylow 19-subgroup, then G does not have an irreducible 19-dimensional complex representation.

Proof. First of all, if G has an irreducible 19-dimensional representation, continuing the notation of the preceding discussion, we must have $\phi(G/P) = \{1, 4, 5, 6, 7, 9, 11, 16, 17\}$:

If not, then $|\phi(G/P)|$ has order 3 or 1. Then let A be a subgroup of order 9 in $\ker(\phi)$. A is abelian and $A \subset \ker(\phi)$ means that the lift $\overline{A}$ of A to G is an abelian subgroup of order 171 in G. Then $[G : \overline{A}] = 12$ so G has no irreducible representation of dimension 19.

Since G has no subgroup of index 2, the Sylow 2-subgroups of G are non-cyclic. If $\ker(\phi) < G/P$ were abelian, then its lift to G would be an abelian subgroup of index 9 in G. This would mean G would have no irreducible 19-dimensional representation. Therefore $\ker(\phi)$ is a nonabelian group of order 12 with noncyclic Sylow 2-subgroups. This means $\ker(\phi) \cong D_{12}$ or A_4.
Now we flesh out a presentation for G to derive the contradiction. Let $< x >$ be the Sylow 19-subgroup of G, and let K be the lift of $\ker(\phi)$ to G, so that $K \cong \ker(\phi) \times P$.
If $\ker(\phi) \cong A_4$, we write $K =< x, a, b, t | x^{19} = 1, xa = ax, xb = bx, xt = tx, a^2 = b^2 = (ab)^2 = t^3 = 1, t^{-1}at = b, t^{-1}bt = ab >$.
Let $z \in G$ be chosen so that $z^{-1}xz = x^4$, and z is a 3-element of G. Then z does not have order 27, since then z^9 would be an element of order 3 in K. The only elements of order 3 in K are the conjugates of t and t^2, which act on $\{a, b, ab\}$ by conjugation via a 3-cycle. So then z would have to act, via a permutation of order 27, on $\{a, b, ab\}$ by conjugation. This is a contradiction because $\{a, b, ab\}$ is too small to admit such an action.
Let τ be a conjugate of t (not necessarily distinct from t) such that $< z, \tau >$ is a Sylow 3-subgroup of G. Since conjugation through elements of $< \tau >$ permutes $\{a, b, ab\}$ via every possible 3-cycle, there is a unique $z' \in \{z, z\tau, z\tau^2\}$ such that z' centralizes $< a, b >$. Then $\tau \in K$ so z' still conjugates x to x^4. $\mathrm{Aut}(A_4) \cong S_4$, with the automorphisms fixing (individually) $a, b, ab \in < a, b, t >$ being the V generated by conjugations through those involutions. Since z' centralizes $< a, b >$, z' has order 9, and V is a 2-group, conjugation through z' has to induce the trivial automorphism on $< a, b, t >$. So we have now established that $G =< a, b, t, x, z' > \cong < a, b, t, x, z' | a^2 = b^2 = (ab)^2 = 1, t^3 = 1, t^{-1}at = b, t^{-1}bt = ab, x^{19} =$

$1, xa = ax, xb = bx, xt = tx, (z')^9 = 1, (z')^{-1}xz' = x^4, z'a = az', z'b = bz', z't = tz' > \cong < x, z' > \times < a, b, t >$. Since G is isomorphic to the Cartesian product of smaller groups and neither of them has an irreducible 19-dimensional representation, the primality of 19 implies G has no such representation itself.

If $\ker(\phi) \cong D_{12}$, we write $K =< x, c, r | x^{19} = 1, xc = cx, xr = rx, c^6 = r^2 = 1, rcr = c^{-1} >$.
Let $z \in G$ be chosen so that $z^{-1}xz = x^4$, and z is a 3-element of G. Then z does not have order 27, since then z^9 would be an element of order 3 in K. The only elements of order 3 in K are c^2 and c^4, which act on $r < c >$ by conjugation via a disjoint product of two 3-cycles. So then z would have to act, via a permutation of order 27, on $r < c >$ by conjugation. This is a contradiction because $r < c >$ is too small to admit such an action.
Since $< c^2 >$ is the unique Sylow 3-subgroup of K, z normalizes it and therefore $< z, c^2 >$ is a Sylow 3-subgroup of G. Since conjugation through elements of $< c^2 >$ permutes $r < c^2 >$ via every possible 3-cycle, there is a unique $z' \in r < c^2 >$ such that z' centralizes $< r, c^2 >$. Since $c^2 \in K$, z' still conjugates x to x^4. Then c^3 is the only central element of order 2 in $< c, r >$, so $< c^3 >$ char $< c, r >$ char$K \lhd G$, so $< c^3 > \lhd G$ and therefore $z'c^3 = c^3z'$. Then z' centralizes $< c^2, r, c^3 > = < r, c >$, and we have now established that $G =< c, r, x, z' > \cong < c, r, x, z' | c^6 = r^2 = 1, rcr = c^{-1}, x^{19} = 1, xc = cx, xr = rx, (z')^9 = 1, z'c = cz', z'r = rz', (z')^{-1}xz' = x^4 > \cong < x, z' > \times < c, r >$. Since G is isomorphic to the Cartesian product of smaller groups and neither of them has an irreducible 19-dimensional representation, G has no such representation itself. □

2 Order 2280

Theorem I. f $|G| = 2280$ and G has a normal Sylow 19-subgroup, then G does not have an irreducible 19-dimensional complex representation.

Proof. $|G/P| = 120$ and G/P has no subgroup of index 2.
If G/P is unsolvable, then $G/P \cong S_5, A_5 \times C_2$, or $SL_2(5)$. Of these, only $SL_2(5)$ lacks a subgroup of index 2. Since $SL_2(5)$ is a perfect group, we get the isomorphism $G \cong P \times G/P$. Since neither P nor G/P has an irreducible representation of dimension 19, G doesn't either.
If G/P is solvable, then it has a Hall $\{2, 5\}$-subgroup H, so that $|H| = 40$. Let C be a cyclic subgroup of order 10 in H. The lift of C to G is a subgroup A of order 190 in G. Since $3 \nmid 190$, every element of A centralizes P and we obtain $A \cong P \times C$. Therefore A is an abelian subgroup of G of index 12, and G cannot have an irreducible 19-dimensional representation in this case. □

3 Order 2736

Theorem I. f $|G| = 2736$ and G has a normal Sylow 19-subgroup, then G does not have an irreducible 19-dimensional complex representation.

Proof. Let Q be a Sylow 2-subgroup of G/P. Let $N \lhd Q$ be chosen so that $|N| = 4$. Then, since N is abelian, Q/N acts in a well-defined way on N by conjugation. Any 2-subgroup of Aut(N) has order 2, while $|Q/N| = 4$. Therefore there is a $y \in Q \setminus N$ such that yN centralizes N and yN has order 2 in Q/N. Then $A = N \cup yN$ is an abelian subgroup of order 8 in G/P. Since P is centralized by any $3'$-element of G, the lift of $A < G/P$ to G is an abelian subgroup of G. The order of this abelian subgroup of G is 152, so its index in G is 18, and G does not have an irreducible representation of dimension 19 in this case. □

4 Order 3040

Theorem I. f $|G| = 3040$ and G has a normal Sylow 19-subgroup, then G does not have an irreducible 19-dimensional complex representation.

Proof. Since $|G/P| = 160$ and $3 \nmid 160$, any element of G centralizes P. Then we get the isomorphism $G \cong P \times G/P$, and neither P nor G/P has an irreducible representation of dimension 19. This means, as before, G cannot have an irreducible representation of dimension 19, and we are done in this case. □

5 Order 3192

Theorem I. f $|G| = 3192$ and G has a normal Sylow 19-subgroup, then G does not have an irreducible 19-dimensional complex representation.

Proof. $|G/P| = 168$ and G/P has no subgroup of index 2. Therefore, $G/P \cong GL_3(2)$, G/P has a normal subgroup of index 3 isomorphic to $C_7 \times Q_8$, or G/P has a normal subgroup of index 3 isomorphic to $AGL_1(\mathbb{F}_8)$.
If $G/P \cong GL_3(2)$, then $GL_3(2)$ is a perfect group so $G \cong P \times GL_3(2)$ and, as before, G has no 19-dimensional irreducible representation.
If G/P has a normal subgroup of index 3 isomorphic to $C_7 \times Q_8$, then, since $3'$-elements of G centralize P, that subgroup of index 3 lifts to a subgroup of G isomorphic to $C_{133} \times Q_8$. But all irreducible representations of the index 3 subgroup $C_{133} \times Q_8$ have dimension ≤ 2, so all irreducible representations of G have dimension ≤ 6, and G has no irreducible representation of dimension 19 in this case.

The remaining possibility is that G/P has a normal subgroup of index 3 isomorphic to $AGL_1(\mathbb{F}_8)$. As before, this lifts to a subgroup N of G for which $N \cong C_{19} \times AGL_1(\mathbb{F}_8)$. Also, $N \lhd G$ and $[G : N] = 3$. Assume that G has an irreducible representation of dimension 19, with χ as its character.
Let $\chi \downarrow N$ be the restriction of χ to N. Let ψ be an irreducible character of N occurring in the decomposition of $\chi \downarrow N$ into irreducible characters of N, and let $\psi \uparrow G$ be the induction of ψ from N to G.
Then $< \chi \downarrow N, \psi >_N = < \chi, \psi \uparrow G >_G$ by Frobenius reciprocity. Since ψ is an irreducible constituent of $\chi \downarrow N$, $< \chi \downarrow N, \psi >_N$ is a positive integer. Therefore $< \chi, \psi \uparrow G >_G$ is a positive integer, being the multiplicity of χ in $\psi \uparrow G$. This multiplicity is bounded from above by $\frac{\psi \uparrow G(1)}{\chi(1)} = \frac{3\psi(1)}{19}$, so $1 \leq \frac{3\psi(1)}{19}$. Therefore $\frac{19}{3} \leq \psi(1) \leq 7$, so $\psi(1) = 7$. Therefore every irreducible constituent of $\chi \downarrow N$ has degree 7, which is a contradiction because $7 \nmid 19$. □