The smallest group with an irreducible representation of dimension 17

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The purpose of this note is to prove that if G is a finite group with an irreducible 17-dimensional complex representation, then $|G| \geq 1751$. By the discussion on the OEIS page for sequence A220470, it suffices to prove that G has no irreducible 17-dimensional representation when $17 \mid |G|$ and $306 \leq |G| \leq 1734$. For all but three of these orders, such a claim is easy to prove using Sylow theorems and Isaacs' Theorem 1.35 (which is the $s = 1$ case of the preliminary theorem below). The three difficult cases are when $|G| = 1632$, $|G| = 612$, and $|G| = 1224$. These are the cases that are examined in detail in this note.

1 One more preliminary result

In addition to the theorems cited there's this:

Theorem I. f G is a finite group with a nontrivial cyclic Sylow 2-subgroup, then G has a subgroup of index 2.

Proof. We write $|G| = 2^s t$, where s is a nonnegative integer and t is an odd integer. In fact, since G has a nontrivial Sylow 2-subgroup, s is positive.
Let x be a generator of a Sylow 2-subgroup of G. Let G act on itself by right-multiplication, and we consider the action of x. All cycles of x have the same length, which is the order of x, namely 2^s. The number of cycles is then $\frac{|G|}{2^s} = t$, so x can be written as the product of $t(2^s - 1)$ transpositions. Since t is odd and s is a positive integer, $t(2^s - 1)$ is odd and the elements of G that act evenly on G form a subgroup N of index 2 in G.

□

2 Order 1632

Theorem I. f $|G| = 1632$, then G does not have an irreducible 17-dimensional complex representation.

Proof. First of all, note that the only divisor d of $\frac{1632}{17} = 96$ satisfying $d \equiv 1 \pmod{17}$ is $d = 1$. Therefore any group of order 1632 has a normal Sylow 17-subgroup P. If G has an irreducible 17-dimensional representation, then G has a subgroup of index 2:

Suppose G does not have a subgroup of index 2. Then G/P does not have a subgroup of index 2, since a subgroup of index 2 in G/P would lift to a subgroup of index 2 in G.
If G/P has no subgroup of index 2, then it is generated by its elements of odd order, so it is generated by its elements of order 3. Their preimages in G all centralize P because $Aut(P)$ is a 2-group. Since P is centralized by a set whose images in G/P generate G/P, P is a central subgroup of G and we can decompose G as $G \cong P \times L$, where $L \cong G/P$.
Now let Q be a Sylow 2-subgroup of L, and let N be a normal subgroup of Q with $|N| = 4$. Then N is abelian, and $|Aut(N)| = 2$ or 6, so a Sylow 2-subgroup of $Aut(N)$ has order 2. Since N is abelian, an element of Q/N acts in a well-defined way by conjugation on N, giving us a homomorphism $\phi : Q/N \to Aut(N)$. Then $|ker(\phi)| = \frac{|Q/N|}{|\phi(Q/N)|} \geq \frac{8}{2} = 4$, so let $y \in Q$ be chosen so that yN is an element of order 2 in $\ker(\phi)$. Then $A = N \cup yN$ is an abelian subgroup of order 8 in L, and G has an abelian subgroup isomorphic to $P \times A$, which has index 12. This proves G does not have an irreducible 17-dimensional representation in this case.

So now let K be a subgroup of index 2 in G, and let χ be the character of an irreducible 17-dimensional representation of G. We consider cases depending on whether or not χ vanishes outside of K. From Theorem 20.12 of James and Liebeck, we learn:
If $\chi(g) \neq 0$ for some $g \in G \setminus K$, then the restriction $\chi \downarrow K$ is irreducible. This is a contradiction because $|K| = 816$ and any group of order 816 has an abelian subgroup of index 16, which rules out the existence of an irreducible character of degree 17.
If $\chi(g) = 0$ for all $g \in G \setminus K$, then the restriction $\chi \downarrow K$ is reducible. Moreover, $\chi = \alpha + \beta$, where $\alpha(1) = \beta(1)$. This gives a contradiction because $\chi(1) = 17$ is odd. □

3 Order 612

Theorem I. f $|G| = 612$, then G does not have an irreducible 17-dimensional complex representation.

Proof. The only divisors d of $\frac{612}{17} = 36$ satisfying $d \equiv 1 \pmod{17}$ are $d = 1$ and 18, so the number of Sylow 17-subgroups of G is either 1 or 18.
If G has just one Sylow 17-subgroup P, then a Sylow 3-subgroup of G/P lifts to an abelian subgroup of index 4 in G, preventing G from having an irreducible representation of dimension 17.
Now we wish to show that assuming G has 18 Sylow 17-subgroups leads to a contradiction:

If G has 18 Sylow 17-subgroups, G acts transitively on them by conjugation. Also, any one of them acts without fixed points on the others, so any one of them acts via a 17-cycle on the others. This means G acts doubly transitively, and therefore primitively, on its Sylow 17-subgroups by conjugation. Then a Sylow 17-subgroup normalizer is a maximal subgroup of index 18 in G. Since 18 is not a power of a prime, this implies G is not solvable.
By how many permutations does G act on its Sylow 17-subgroups? Since G acts doubly transitively, this number is a multiple of $18 \cdot 17 = 306$. But any group of order 306 is solvable, so this number must be 612 and G must act faithfully. So we regard G as a permutation group on 18 points.

A Sylow 17-subgroup normalizer in G is a group of order 34 that acts faithfully on 17 points, so it must be dihedral. Then any 2-point stabilizer in G has a unique element of order 2, which is a disjoint product of 8 2-cycles. Also, no element of G, except the identity, has more than 2 fixed points.
G has no derangements of order 2: a derangement of order 2 would be the product of 9 transpositions, so it would be an odd permutation. But G has no subgroup of index 2 because G is unsolvable but all groups of order 306 are solvable.
Since G has no subgroup of index 2, G can't have a cyclic Sylow 2-subgroup. Therefore G has no elements of order 4.
Then the number of solutions to $x^4 = 1$ in G is $1 + \binom{18}{2} = 154$, but 154 is not a multiple of 4. This contradicts Theorem 9.1.2 from Hall, and the contradiction is established.

□

In fact, the above reasoning shows that any group of order 612 has a unique, and therefore normal, Sylow 17-subgroup. Since all groups of order 36 are solvable, this establishes that all groups of order 612 are solvable.

4 Order 1224

Theorem I. f $|G| = 1224$, then G does not have an irreducible 17-dimensional complex representation.

Proof. The only divisors d of $\frac{1224}{17} = 72$ satisfying $d \equiv 1 \pmod{17}$ are $d = 1$ and 18, so the number of Sylow 17-subgroups of G is either 1 or 18.
If G has just one Sylow 17-subgroup P, then a Sylow 3-subgroup of G/P lifts to an abelian subgroup of index 8 in G, preventing G from having an irreducible representation of dimension 17.
Now we wish to show that assuming G has 18 Sylow 17-subgroups leads to a contradiction:

If G has 18 Sylow 17-subgroups, G acts doubly transitively on them by conjugation, as before. As before, this implies G is unsolvable.
By how many permutations does G act on its Sylow 17-subgroups? Since G acts doubly transitively, this number is a multiple of 306. Also, this number must divide $|G| = 1224$, so it must be 306, 612, or 1224. It cannot be 306 or 612 because all groups of those orders are solvable. Therefore this number must be 1224 and G must act faithfully. So we regard G as a permutation group on 18 points.
A Sylow 17-subgroup normalizer in G is a group of order 68 that acts faithfully on 17 points, so it must be isomorphic to a subgroup of index 4 in $AGL_1(17)$. Then any 2-point stabilizer in G has a unique element of order 2 (a disjoint product of 8 2-cycles), and two elements of order 4 (each of which is a disjoint product of 4 4-cycles). Also, no element of G, except the identity, has more than 2 fixed points.
G has no derangements of order 2, as before.
G has no elements of order 8, since then G would have cyclic Sylow 2-subgroups. If G had a subgroup of index 2, that subgroup would have order 612 and therefore be solvable, contradicting the unsolvability of G.

G has no derangements of order 4:
Every cycle of a derangement x of order 4 has length 2 or 4. The points in a 4-cycle of x are those which are in 2-cycles of x^2. Since every element of order 2 in G is a disjoint product of 8 2-cycles, there are 16 such points. Then the only possible cycle structure for a derangement of order 4 in G is (4,4,4,4,2). But any permutation with that cycle structure is an odd permutation, whose presence in G would contradict the unsolvability of G.
Therefore all elements of order 2 or 4 in G have exactly 2 fixed points. Then the total number of solutions to $x^8 = 1$ in G is $1 + \binom{18}{2} + 2\binom{18}{2} = 460$, which is not a multiple of 8. This contradicts Theorem 9.1.2 of the Hall reference, and establishes the desired contradiction.

$\square$