A family of integer sequences

Robert K. Moniot Fordham University

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1 Introduction

This article presents a set of properties of the family of integer sequences $[a_n(\rho)]$ with ordinary generating function, parameterized by integer ρ ,

$$\alpha(x) = \frac{1}{(1-x)(1-\rho x + x^2)}.$$
(1)

A list of 23 sequences that belong to this family is given in [5] at sequence <u>A212336</u>, with that sequence itself being the 24th. (Such sequence A-numbers will be used later on without citing *OEIS*). That list parameterizes them by $k = \rho + 1$, but for this article using ρ makes most expressions simpler.

In what follows, we will write a_n for $a_n(\rho)$.

For some of these sequences, the entries on OEIS use an offset such that $a_0 = 0$, while for others $a_0 = 1$. All of the sequences are compatible with having 0 as the element before 1, and can be extended to negative n in a manner described in section 3. For this article, we adopt the convention $a_0 = 1$.

Section 14 lists some especially notable members of this family.

2 List of shared properties

The sequences in this family, parameterized by ρ , have, in addition to (1), the following properties. Proofs follow in later sections.

1. Non-homogeneous linear recurrence:

$$\begin{aligned}
a_{-1} &= 0, \\
a_{0} &= 1, \\
a_{n} &= \rho a_{n-1} - a_{n-2} + 1, \quad n \in \mathbb{Z}.
\end{aligned}$$
(2)

2. Homogeneous linear recurrence:

$$\begin{aligned}
a_{-2} &= 0, \\
a_{-1} &= 0, \\
a_{0} &= 1, \\
a_{n} &= (\rho+1)(a_{n-1}-a_{n-2}) + a_{n-3}, \quad n \in \mathbb{Z}.
\end{aligned}$$
(3)

3. Nonlinear 3-term identity:

$$a_{n+1}a_{n-1} = a_n(a_n - 1), \quad n \in \mathbb{Z}.$$
 (4)

4. Partial sums of Chebyshev polynomials:

$$a_n = \sum_{i=0}^n S_i(\rho), \quad n = 0, 1, 2, \dots,$$
 (5)

where $S_n(\rho) = U_n(\rho/2)$ are the Chebyshev polynomials of the second kind (A049310).

5. Difference of Chebyshev polynomials: if $\rho \neq 2$,

$$a_n = \frac{1}{\rho - 2} \left(S_{n+1}(\rho) - S_n(\rho) - 1 \right), \quad n \in \mathbb{Z}.$$
 (6)

For n < 0, use $S_{-1}(\rho) = 0$ and $S_{-n}(\rho) = -S_{n-2}(\rho)$ for $n \ge 2$.

6. Diophantine equation: If $x = a_n$ and $y = a_{n+1}$ are successive terms in the sequence, $n \in \mathbb{Z}$, then

$$x^{2} - x + y^{2} - y - \rho x y = 0.$$
(7)

These are all the integer solutions of the equation except when $\rho = 3$ or $\rho = 4$, for each of which an additional sequence of solutions exists, given in Appendix A.

7. Related binary quadratic form: define the sequence $b_n = (\rho - 2)a_n + 1$. If $u = b_n$ and $v = b_{n+1}$ are successive terms in the sequence, $n \in \mathbb{Z}$, then

$$u^{2} + v^{2} - \rho u v = -(\rho - 2).$$
(8)

8. Triangular number identity: if $\rho \ge 2$, then if $x = a_n$ and $y = a_{n+1}$ are successive terms in the sequence, $n \ge 0$,

$$\frac{T(x-1) + T(y-1)}{T(x+y-1)} = \frac{\rho}{\rho+2},\tag{9}$$

where T(i) is the *i*-th triangular number.

9. One-term formula giving squares:

$$(\rho^2 - 4)a_n^2 + 2(\rho + 2)a_n + 1 = (a_{n+1} - a_{n-1})^2, \quad n \in \mathbb{Z}.$$
 (10)

10. Two-term formula giving squares:

$$4a_{n+1}a_{n-1} + 1 = (2a_n - 1)^2, \quad n \in \mathbb{Z}.$$
(11)

11. Closed-form formulas: these are simplest if n = 0 yields a value of 0. Rather than change the offset, we give formulas for a_{n-1} .

(a) If
$$\rho \neq \pm 2$$
:
$$a_{n-1} = \frac{1}{2-\rho} + c_1 r_1^n + c_2 r_2^n, \quad n \in \mathbb{Z}$$
(12)

where

$$r_{1} = \frac{1}{2} \left(\rho - \sqrt{\rho^{2} - 4} \right),$$

$$r_{2} = \frac{1}{2} \left(\rho + \sqrt{\rho^{2} - 4} \right),$$
(13)

and

$$c_{1} = \frac{1}{2} \left(\frac{1}{\rho - 2} - \frac{1}{\sqrt{\rho^{2} - 4}} \right),$$

$$c_{2} = \frac{1}{2} \left(\frac{1}{\rho - 2} + \frac{1}{\sqrt{\rho^{2} - 4}} \right).$$
(14)

(b) If $\rho = 2$:

$$a_{n-1} = \frac{n(n+1)}{2}, \quad n \in \mathbb{Z}.$$
 (15)

(c) If $\rho = -2$:

$$a_{n-1} = \frac{1}{4} \left(1 - (-1)^n (2n+1) \right), \quad n \in \mathbb{Z}.$$
 (16)

We now give the proofs of the above statements.

3 Non-homogeneous linear recurrence

The proof of (2) proceeds by showing that the sequence the recurrence generates has the ordinary generating function (1). By definition,

$$\alpha(x) = \sum_{n \ge 0} a_n x^n.$$

Using the recurrence (2),

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = \sum_{n\geq 0} (\rho a_n - a_{n-1} + 1)x^{n+1};$$

$$\alpha(x) - a_0 = \rho x \sum_{n\geq 0} a_n x^n - a_{-1}x - x^2 \sum_{n\geq 0} a_n x^n + x \sum_{n\geq 0} x^n$$

$$= \rho x \alpha(x) - a_{-1}x - x^2 \alpha(x) + \frac{x}{1-x}.$$

Setting $a_{-1} = 0$ and $a_0 = 1$ and solving for $\alpha(x)$ gives (1).

The recurrence (2) allows the sequence to be extended to negative n by using it to solve for a_{n-2} from a_{n-1} and a_n . This yields $a_{-2} = 0$ and $a_{-n} = a_{n-3}$ for n > 0.

Observe that $a_1 = \rho + 1$. It is interesting that $a_2 = \rho(\rho + 1)$, which is twice a triangular number.

4 Homogeneous linear recurrence

Formula (3) follows at once by subtracting (2) from itself offset by 1 step.

5 Nonlinear 3-term identity

The identity (4) can be shown to hold by induction. The triple $(a_{-1}, a_0, a_1) = (0, 1, \rho + 1)$ satisfies (4). Now assume

$$a_{n+1}a_{n-1} - a_n(a_n - 1) = 0$$

for some $n \ge 0$. Using (2),

$$a_{n+2}a_n - a_{n+1}(a_{n+1} - 1) = (\rho a_{n+1} - a_n + 1)a_n - a_{n+1}(\rho a_n - a_{n-1} + 1 - 1)$$

= $\rho (a_{n+1}a_n - a_{n+1}a_n) - a_n(a_n - 1) + a_{n+1}a_{n-1}$
= 0.

6 Partial sum of Chebyshev polynomials

The ordinary generating function of the Chebyshev polynomials is [4]

$$\sum_{n=0}^{\infty} S_n(\rho) x^n = \frac{1}{1 - \rho x + x^2}.$$
(17)

Also, if $\alpha(x)$ is the generating function for a sequence $[a_n]$, then

$$\frac{1}{1-x} \alpha(x)$$

is the generating function for

$$b_n = \sum_{i=0}^n a_i.$$

Therefore (5) follows immediately from (1).

7 Difference of Chebyshev polynomials

We prove (6) by induction, using the following recurrence for the Chebyshev polynomials [4]:

$$S_{0}(\rho) = 1,
S_{1}(\rho) = \rho,
S_{n}(\rho) = \rho S_{n-1}(\rho) - S_{n-2}(\rho), \quad n > 0.$$
(18)

Both $a_{-1} = 0$ and $a_0 = 1$ satisfy (6), using $S_{-1}(\rho) = 0$. Now assume that (6) holds up to some n > 0. Then using (2),

$$\begin{aligned} a_{n+1} &= \rho \, a_n - a_{n-1} + 1 \\ &= \rho \, \frac{S_{n+1}(\rho) - S_n(\rho) - 1}{\rho - 2} - \frac{S_n(\rho) - S_{n-1}(\rho) - 1}{\rho - 2} + 1 \\ &= \frac{(\rho \, S_{n+1}(\rho) - S_n(\rho)) - (\rho \, S_n(\rho) - S_{n-1}(\rho)) - (\rho - 1) + (\rho - 2)}{\rho - 2} \\ &= \frac{1}{\rho - 2} \left(S_{n+2}(\rho) - S_{n+1}(\rho) - 1 \right). \end{aligned}$$

8 Diophantine equation

We prove (7) by induction. For n = -1, x = 0 and y = 1. These satisfy (7). Now for the inductive step, assume (rewriting (7) in a more convenient form)

$$a_n(a_n - 1) + a_{n+1}(a_{n+1} - 1) - \rho \, a_n a_{n+1} = 0$$

for some $n \ge -1$. Then using (2),

$$a_{n+2}(a_{n+2}-1) = (\rho a_{n+1} - a_n + 1)(\rho a_{n+1} - a_n)$$

= $\rho^2 a_{n+1}^2 - 2\rho a_{n+1} a_n + \rho a_{n+1} + a_n(a_n - 1)$

and

$$\rho a_{n+1}a_{n+2} = \rho a_{n+1}(\rho a_{n+1} - a_n + 1)$$

= $\rho^2 a_{n+1}^2 - \rho a_{n+1}a_n + \rho a_{n+1}.$

Hence

$$a_{n+1}(a_{n+1}-1) + a_{n+2}(a_{n+2}-1) - \rho \, a_{n+1}a_{n+2} = a_n(a_n-1) + a_{n+1}(a_{n+1}-1) - \rho \, a_n a_{n+1} = 0.$$

8.1 Completeness

The sequence $[a_n]$ given by (2) essentially provides all the solutions of (7), once a few details are accounted for. The proof and details are deferred to Appendix A.

8.2 Connection to a probability inversion problem

Observe that (7) rearranges as

$$\frac{2}{\rho+2} = \frac{2xy}{(x+y)(x+y-1)},\tag{19}$$

provided $\rho \neq -2$ and $x + y \neq 0$ or 1. (We retain the 2 on both sides for the sake of the following interpretation.) If $\rho \geq 0$, then $0 \leq 2/(\rho + 2) \leq 1$. For solutions having $x \geq 0$, $y \geq 0$, and $x + y \geq 2$, (19) can be interpreted as the probability of drawing two different-colored socks at random from a drawer containing xsocks of one color and y socks of another color. Hence the solutions of the Diophantine equation (7) solve the problem of finding numbers x and y that yield the probability given by a specific choice of ρ .

Of course, if ρ is integer, not all ratios corresponding to arbitrary choices of x and y in (19) are represented. Allowing ρ to be fractional, any rational probability can be represented. The problem of finding x and y that satisfy (19) in those cases is more complicated [3].

9 Related binary quadratic form

Setting $u = (\rho - 2)x + 1$, $v = (\rho - 2)y + 1$, where (x, y) is a solution of (7),

$$u^{2} + v^{2} - \rho u v = ((\rho - 2)x + 1)^{2} + ((\rho - 2)y + 1)^{2}$$

-\rho ((\rho - 2)x + 1) ((\rho - 2)y + 1)
= (\rho - 2) ((\rho - 2)x^{2} + 2x + (\rho - 2)y^{2} + 2y
-\rho ((\rho - 2)x y + x + y)) + 2 - \rho
= (\rho - 2)^{2} (x^{2} - x + y^{2} - y - \rho x y) + 2 - \rho
= -(\rho - 2).

This proves that (u, v) satisfies (8).

These solutions form a family of integer sequences

$$b_n = (\rho - 2)a_n + 1, \tag{20}$$

where, as for a_n , we write b_n for $b_n(\rho)$. If $\rho = 2$, $b_n = 1$, $\forall n$, which satisfy (8) trivially. In this case, (20) is not invertible, and other solutions of (8) exist, but nonetheless the sequence $[b_n]$ is defined.

The members of this family of sequences obey the recurrence

$$b_{-1} = 1, b_0 = \rho - 1, b_n = \rho b_{n-1} - b_{n-2}, \quad n > 0.$$
(21)

These sequences satisfy

$$b_n = S_{n+1}(\rho) - S_n(\rho),$$
(22)

with $S_n(\rho)$ as defined in (18). Appendix B lists OEIS sequence A-numbers for some members of this family.

10 Triangular number identity

Subtracting each side of (19) from 1, we obtain (9):

$$\frac{\rho}{\rho+2} = \frac{x(x-1) + y(y-1)}{(x+y)(x+y-1)} = \frac{T(x-1) + T(y-1)}{T(x+y-1)}$$

where T(n) = n(n+1)/2 is the *n*-th triangular number.

11 One-term formula giving squares

Using (2),

$$(a_{n+1} - a_{n-1})^2 = (\rho a_n - 2a_{n-1} + 1)^2$$

= $\rho^2 a_n^2 + 4a_{n-1}^2 - 4\rho a_n a_{n-1} + 2\rho a_n - 4a_{n-1} + 1$
= $(\rho^2 - 4)a_n^2 + 2(\rho + 2)a_n + 1 +$
 $4 (a_n^2 - a_n + a_{n-1}^2 - \rho a_n a_{n-1} - a_{n-1})$

The final term in parentheses is 0 using the fact that (a_{n-1}, a_n) is a solution of (7). This proves (10).

The sequence of terms that are squared is given by

$$a_{n+1}(\rho) - a_{n-1}(\rho) = S_{n+1}(\rho) + S_n(\rho)$$
(23)

$$= S_{2n+2}(\sqrt{\rho+2})$$
 (24)

$$= (-1)^{n+1} b_n(-\rho), \qquad (25)$$

with $S_n(\rho)$ as in (18) and $b_n(\rho)$ as in (20). (Proofs omitted.) Appendix B lists OEIS sequence A-numbers for some of the sequences in this family.

12 Two-term formula giving squares

Using (4),

$$4a_{n+1}a_{n-1} + 1 = 4a_n(a_n - 1) + 1$$

= $(2a_n - 1)^2$

This proves (11). The sequence of terms that are squared is given by $2a_n(\rho) - 1$. Appendix B lists OEIS sequence A-numbers for some of the sequences in this family.

13 Closed-form formulas

We derive the closed-form formulas from the non-homogeneous recurrence formula (2). The associated homogeneous recurrence relation of (2) is

$$a_n - \rho \, a_{n-1} + a_{n-2} = 0,$$

whose characteristic equation is

$$r^2 - \rho r + 1 = 0. \tag{26}$$

The roots of this equation are as in (13). They are distinct if $\rho \neq \pm 2$. Treating each of the cases in turn:

1. If $\rho \neq \pm 2$, the particular solution of (2) is of the form of a constant, $a_n = c_0, \forall n$. Inserting this into (2) and solving yields

$$c_0 = \frac{1}{2-\rho}.$$

Combining the particular and the homogeneous solutions, the closed-form solution is of the form

$$c_0 + c_1 r_1^n + c_2 r_2^n, (27)$$

where the constants c_1 and c_2 are determined by the initial conditions. Equating (27) to 0 for n = 0 and 1 for n = 1, we obtain (14), proving (12).

- 2. If $\rho = 2$, the formula (15) satisfies (2) and agrees with the initial value $a_0 = 1$.
- 3. If $\rho = -2$, the particular solution of (2) is of the form of a constant, $a_n = c_0, \forall n$. Inserting this into (2) with $\rho = -2$ yields $c_0 = 1/4$. The roots (13) are $r_1 = r_2 = \rho/2 = -1$. Combining the particular and homogeneous solutions, the closed-form solution formula is of the form

$$c_0 + (-1)^n c_1 + (-1)^n n c_2. (28)$$

Equating (28) to 0 for n = 0 and 1 for n = 1 yields $c_1 = -1/4$, $c_2 = -1/2$, which gives (16).

14 Notable sequences

The following notable sequences are documented on OEIS at their respective A-numbers.

- For $\rho = -2$ $(a_n = \underline{A001057}(n+1))$, the sequence including 0 is the canonical enumeration of the integers, interleaving the positive and negative integers.
- For $\rho = 2$ $(a_n = \underline{A000217}(n+1))$, the sequence is the triangular numbers.
- For $\rho = 3$ $(a_n = \underline{A027941}(n+1))$, the related sequence defined in (20) is $b_n = a_n + 1 = \underline{A001519}(n+2)$, a bisection of the Fibonacci numbers: $b_n = F_{2n+3}$ with $F_n = \underline{A000045}(n)$ the Fibonacci numbers. The values b_n are a subset of the Markov numbers $\underline{A002559}$: $(1, b_n, b_{n+1})$ is a Markov triple.

• For $\rho = 14$ $(a_n = \underline{A076139}(n+1))$, the sequence consists of triangular numbers that are one-third of another triangular number: $a_n = T_m$ such that $3T_m = T_k$ for some k, with $T_n = n(n+1)/2$ the triangular numbers, $T_n = \underline{A000217}(n)$.

15 Acknowledgements

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Appendix A Completeness of solutions of (7)

We show that for any integer ρ , the sequence of values $[a_n]$ generated by the recurrence (2) provides the complete set of solutions of (7), except for two values of ρ for which additional solutions exist that are generated using the same recurrence formula with different starting values.

In order for the sequence of solutions to be complete, elements for n < 0 need to be included, in order to include the solutions (0,0) and (0,1), as well as those for which x > y, which by symmetry are also solutions.

To show that no solutions are missed, we need to recast (7) in a form for which the complete set of solutions can be readily found. Let

$$t = y + x, \quad s = y - x.$$
 (29)

Then (7) becomes

$$(\rho - 2)t^2 + 4t - (\rho + 2)s^2 = 0.$$
(30)

If $\rho \neq 2$, we can complete the square on t in (30). Setting

$$r = (\rho - 2)t + 2, \quad D = \rho^2 - 4,$$
(31)

we obtain

$$r^2 - Ds^2 = 4. (32)$$

(If ρ is even, this equation can be divided through by 4 to obtain the Pell equation. For the sake of uniformity in the treatment, we leave it in this form.)

We note in passing that the Chebyshev polynomials satisfy the Pell equation

$$T_n(x)^2 - (x^2 - 1)U_{n-1}(x)^2 = 1,$$

where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first and second kind, respectively [1]. Therefore

$$r_n = R_n(\rho), \quad s_n = S_{n-1}(\rho), \quad n \in \mathbb{Z},$$
(33)

are solutions of (32), where $R_n(\rho) = 2T_n(\rho/2)$ and $S_n(\rho) = U_n(\rho/2)$. The OEIS A-numbers for the sequences $[r_n]$ and $[s_n]$ for some values of ρ are given in Appendix B.

We now treat the different cases for different values of ρ in turn.

A.1 The case with $\rho = 2$

If $\rho = 2$, (30) reduces to the parabola $t = s^2$. Solving

$$\begin{array}{rcl} y+x &=& s^2,\\ y-x &=& s, \end{array}$$

we obtain x = s(s-1)/2 and y = s(s+1)/2. Identifying s with n, this result is the same as the closed-form solution (15). Hence for $\rho = 2$ the sequence is the complete set of solutions.

A.2 The cases with $\rho = 0, \pm 1$

If $|\rho| < 2$, D < 0 and (7) is a small ellipse, whose integer solution points can be found by inspection. The sequence $[a_n]$ repeats.

- $\rho = 1$: the sequence is the period-6 repeat [0, 1, 2, 2, 1, 0, ...], generating the six solution points (0, 1), (1, 2), (2, 2), (2, 1), (1, 0), (0, 0).
- $\rho = 0$: the sequence is the period-4 repeat $[0, 1, 1, 0, \ldots]$, generating the four solution points (0, 1), (1, 1), (1, 0), (0, 0).
- $\rho = -1$: the sequence is the period-3 repeat [0, 1, 0, ...], generating the three solution points (0, 1), (1, 0), (0, 0).

A.3 The case with $\rho = -2$

If $\rho = -2$, (30) reduces to t(t-1) = 0, i.e., y = -x or y = 1-x. In this case, (7) is a pair of straight lines of slope -1, passing through (0,0) and (0,1) respectively. The recurrence (2) with $\rho = -2$ takes the point (n, -n) to (-n, n+1) and (-n, n+1) to (n+1, -(n+1)). Therefore, starting with the solution (0,0) it generates half of the points on these two lines. Running the recurrence in reverse generates the rest, those of the form (-n, n) and (n+1, -n).

A.4 The cases with $\rho > 2$ or $\rho < -2$

If $|\rho| > 2$, D > 0 and (7) is a hyperbola. For these cases, $D = \rho^2 - 4$ cannot be square. Therefore all solutions of (32) are given by

$$\frac{r+s\sqrt{D}}{2} = \pm \left(\frac{r_1 + s_1\sqrt{D}}{2}\right)^n, \quad n = 0, \pm 1, \pm 2, \dots,$$
(34)

where (r_1, s_1) is the solution with $r_1 > 0$, $s_1 > 0$ for which $r_1 + s_1\sqrt{D}$ is least [2, Theorem 4.4]. Clearly $r_1 = |\rho|$ and $s_1 = 1$. We can drop the absolute value since for $\rho < 0$ it only changes the sign of r and s. Inserting $(r_1, s_1) = (\rho, 1)$ into (34) and using the positive sign in front, for $n \ge 0$, we can write (r, s) on the left in (34) as (r_n, s_n) and obtain a recurrence formula:

$$r_{n} = \frac{1}{2} \left(\rho r_{n-1} + D s_{n-1} \right),$$

$$s_{n} = \frac{1}{2} \left(r_{n-1} + \rho s_{n-1} \right).$$
(35)

Converting this to a recurrence formula for x and y using (31),

$$(\rho - 2)(y_n + x_n) + 2 = \frac{1}{2} \left(\rho \left((\rho - 2)(y_{n-1} + x_{n-1}) + 2 \right) + (\rho^2 - 4)(y_{n-1} - x_{n-1}) \right),$$

$$y_n - x_n = \frac{1}{2} \left((\rho - 2)(y_{n-1} + x_{n-1}) + \rho \left(y_{n-1} - s_{n-1} \right) \right).$$

which simplifies to

$$(\rho - 2)(y_n + x_n) = (\rho - 2)(1 + (\rho + 1)y_{n-1} - x_{n-1}),$$

 $y_n - x_n = (\rho - 1)y_{n-1} - x_{n-1} + 1.$

Cancelling $\rho - 2$, which is nonzero, and solving, we obtain

$$\begin{array}{rcl} x_n & = & y_{n-1}, \\ y_n & = & \rho \, y_{n-1} - x_{n-1} + 1, \end{array}$$

which is the recurrence formula (2) with $x_n = a_{n-1}$ and $y_n = a_n$.

The solution $(r_0, s_0) = (2, 0)$ is obtained with n = 0 in (34). It maps to (x, y) = (0, 0). Therefore, using the recurrence (35) with $(r_0, s_0) = (2, 0)$ gives a sequence of (r_n, s_n) values that map to the same sequence of a_n values as using (2) starting from $(a_{-2}, a_{-1}) = (0, 0)$. Negative values of s correspond to negative values of n in (34), and simply interchange x and y. These solutions are given by running (2) in reverse.

Finally, we need to consider values of r with the opposite sign, corresponding to using the negative sign in front of the right-side expression in (34). For $\rho > 2$ these map to the branch of the hyperbola in the third quadrant of (x, y)-space. The solution $(r_0, s_0) = (-2, 0)$ maps to the vertex $(x, y) = (\frac{-2}{\rho-2}, \frac{-2}{\rho-2})$. This is an integer solution only for $\rho \in \{0, 1, 3, 4\}$. The values $\rho = 0$ and 1 are in the elliptical regime treated earlier. For the values $\rho = 3$ and 4 the vertex solutions are (x, y) = (-2, -2) and (-1, -1) respectively. Using alternate starting values $a_{-2} = a_{-1} = -2$ or $a_{-2} = a_{-1} = -1$ in the recurrence (2) yields the same sequences of solutions as using (35) with $(r_0, s_0) = (-2, 0)$ (equivalent to (34) with $n \ge 0$ and the minus sign in front). These negative sequences do not match OEIS sequences, but changing the sign they do. For $\rho = 3$, the alternate $a_n = -\underline{A032908}(n+2)$ and for $\rho = 4$ it is $a_n = -\underline{A101879}(n+2)$. Comments on these sequences by Robert Israel, Aug 26 2015 and Andrey Vyshnevyy, Sep 18 2015, respectively, note that consecutive terms of these sequences provide all positive integer pairs for which K = (a + 1)/b + (b + 1)/a is integer. This is equivalent to (7) with x = -a, y = -b, and $\rho = K$.

If $\rho < -2$, the vertex of the hyperbola in (x, y)-space that is not coincident with the origin is never on integer coordinates. (For these values of ρ , the two branches of the hyperbola lie in the second and fourth quadrants, except for a short piece of one branch in the first quadrant between (0, 1) and (1, 0), with the other branch passing through (0, 0).) Therefore the only integer solutions of (7) for $\rho < -2$ are those given by the plus sign in front of the right-side expression in (34), equivalent to the recurrence (2) started with (0, 0).

This concludes the proof of completeness.

Appendix B Related sequences

Here we list OEIS A-numbers for the sequences $[a_n]$ defined by (1), as well as for some related sequences. They are parameterized by ρ . The sequences $[b_n]$ given by (20) give solutions of the related binary quadratic form (8). The sequences $[r_n]$ and $[s_n]$ are the non-negative solutions of (32), given by (33).

The OEIS A-numbers for these sequences are listed in Table 1. The table includes $\rho = -1$ to 22, which are the values of ρ for the sequences listed at <u>A212336</u> and that sequence itself.

The offsets between the sequences as defined here with $a_0 = 1$ and the OEIS sequences are shown: e.g., for $\rho = 2$, the OEIS sequence for a_n starts with 0.

Table 2 gives OEIS A-numbers for the sequences of values whose squares are produced by the one-term formula (10) and the two-term formula (11). The index of the former is offset to match the offsets of the OEIS sequences.

References

- [1] Demeyer, Jeroen (2007). Diophantine Sets over Polynomial Rings and Hilbert's Tenth Problem for Function Fields. PhD Thesis, Universiteit Gent. Archived copy available.
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ρ	a_n	b_n	r_n	s_n
-1	$\underline{A079978}(n)$	A131713(n+1)		$\underline{A102283}(n)$
0	$\underline{A133872}(n)$	A087960(n+1)		$\underline{A056594}(n-1)$
1	$\underline{A164965}(n)$	A010892(n+2)	A057079(n+1)	$\underline{A010892}(n+2)$
2	A000217(n+1)	$\underline{A000012}(n)$	$\underline{A007395}(n)$	$\underline{A001477}(n)$
3	$A027941(n+1)^{-1}$	A001519(n+2)	$\underline{A005248}(n)$	$\underline{A001906}(n)$
4	<u>A061278</u> $(n+1)^{-1}$	A079935(n+1)	$\underline{A003500}(n)$	$\underline{A001353}(n)$
5	$\underline{A089817}(n)$	A004253(n+1)	$\underline{A003501}(n)$	$\underline{A004254}(n)$
6	A053142(n+1)	A001653(n+1)	$\underline{A003499}(n)$	$\underline{A001109}(n)$
7	$\underline{A092521}(n)$	A049685(n+1)	$\underline{A056854}(n)$	$\underline{A004187}(n)$
8	$\underline{A076765}(n)$	A070997(n+1)	$\underline{A086903}(n)$	$\underline{A001090}(n)$
9	$\underline{A092420}(n)$	A070998(n+1)	$\underline{A056918}(n)$	$\underline{A018913}(n)$
10	$\underline{A097784}(n)$	A072256(n+2)	$\underline{A087799}(n)$	$\underline{A004189}(n)$
11	$\underline{A097826}(n)$	A078922(n+1)	$\underline{A057076}(n)$	A004190(n-1)
12	$\underline{A097828}(n)$	A077417(n+1)	$\underline{A087800}(n)$	A004191(n-1)
13	$\underline{A097828}(n)$	$A085260(n+1)^{-2}$	$\underline{A078363}(n)$	A078362(n-1)
14	A076139(n+1)	A001570(n+1)	$\underline{A067902}(n)$	$\underline{A007655}(n)$
15	$\underline{A097829}(n)$	A160682(n+1)	$\underline{A078365}(n)$	A078364(n-1)
16	$\underline{A097830}(n)$	A157456(n+1)	$\underline{A090727}(n)$	A077412(n-1)
17	$\underline{A097831}(n)$	A161595(n+1)	$\underline{A078367}(n)$	<u>A078366</u> $(n-1)$
18	A049664(n+1)	A007805(n+1)	$\underline{A087215}(n)$	$\underline{A049660}(n)$
19	$\underline{A097832}(n)$	—	$\underline{A078369}(n)$	A078368(n-1)
20	$\underline{A097833}(n)$	A075839(n+1)	$\underline{A090728}(n)$	$\underline{A075843}(n)$
21	$\underline{A212335}(n)$		$\underline{A090729}(n)$	$\underline{A092499}(n)$
22	$\underline{A212336}(n)$	A157014(n+1)	$\underline{A090730}(n)$	A077421(n-1)

Table 1: OEIS A-numbers for the sequences $[a_n]$ having the ordinary generating function (1), for $[b_n]$ as defined by (20), and for $[r_n]$ and $[s_n]$ as defined by (33).

¹ As noted in Appendix A, $\rho = 3$ and $\rho = 4$ each have an additional sequence of integer solutions of (7) on the negative branch of the hyperbola. The alternate solutions are: -<u>A032908</u>(n + 2) for $\rho = 3$ and and -<u>A101879</u>(n + 2) for $\rho = 4$.

² Based on sequence description "Ratio-determined insertion sequence I(0.0833344)," identification of this sequence with $b_n(13)$ as defined here is conjectural. See comment by M. F. Hasler Nov 05 2018.

Table 2: OEIS A-numbers for the sequences $[a_n - a_{n-2}]$ and $[2a_n - 1]$ for $\rho = -1$ to 22. These sequences are the values whose squares are produced by the formulas (10) and (11) respectively.

ρ	$a_n - a_{n-2}$	$2a_n - 1$
-1	$\underline{A057078}(n)$	
0	$\underline{A057077}(n)$	$\underline{A057077}(n)$
1	A057079(n)	$-\underline{A130778}(n)$
2	A005408(n)	A028387(n)
3	A002878(n)	_
4	A001834(n)	
5	A030221(n)	
6	A002315(n)	
7	A033890(n)	
8	$\underline{A057080}(n)$	
9	A057081(n)	
10	$\underline{A054320}(n)$	
11	A097783(n)	
12	A077416(n)	
13	A126866(n)	
14	A028230(n)	
15	$\underline{A161591}(n)$	
16	$\underline{A159678}(n)$	
17	$\underline{A161599}(n)$	
18	$\underline{A049629}(n)$	
19		
20	$\underline{A083043}(n)$	
21		
22	$\underline{A133283}(n)$	

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