# A family of integer sequences 

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## 1 Introduction

This article presents a set of properties of the family of integer sequences $\left[a_{n}(\rho)\right]$ with ordinary generating function, parameterized by integer $\rho$,

$$
\begin{equation*}
\alpha(x)=\frac{1}{(1-x)\left(1-\rho x+x^{2}\right)} . \tag{1}
\end{equation*}
$$

A list of 23 sequences that belong to this family is given in [5] at sequence A212336, with that sequence itself being the 24th. (Such sequence A-numbers will be used later on without citing $O E I S$ ). That list parameterizes them by $k=\rho+1$, but for this article using $\rho$ makes most expressions simpler.

In what follows, we will write $a_{n}$ for $a_{n}(\rho)$.
For some of these sequences, the entries on OEIS use an offset such that $a_{0}=0$, while for others $a_{0}=1$. All of the sequences are compatible with having 0 as the element before 1 , and can be extended to negative $n$ in a manner described in section 3. For this article, we adopt the convention $a_{0}=1$.

Section 14 lists some especially notable members of this family.

## 2 List of shared properties

The sequences in this family, parameterized by $\rho$, have, in addition to (1), the following properties. Proofs follow in later sections.

1. Non-homogeneous linear recurrence:

$$
\begin{align*}
a_{-1} & =0 \\
a_{0} & =1  \tag{2}\\
a_{n} & =\rho a_{n-1}-a_{n-2}+1, \quad n \in \mathbb{Z}
\end{align*}
$$

2. Homogeneous linear recurrence:

$$
\begin{align*}
a_{-2} & =0 \\
a_{-1} & =0  \tag{3}\\
a_{0} & =1 \\
a_{n} & =(\rho+1)\left(a_{n-1}-a_{n-2}\right)+a_{n-3}, \quad n \in \mathbb{Z}
\end{align*}
$$

3. Nonlinear 3-term identity:

$$
\begin{equation*}
a_{n+1} a_{n-1}=a_{n}\left(a_{n}-1\right), \quad n \in \mathbb{Z} \tag{4}
\end{equation*}
$$

4. Partial sums of Chebyshev polynomials:

$$
\begin{equation*}
a_{n}=\sum_{i=0}^{n} S_{i}(\rho), \quad n=0,1,2, \ldots \tag{5}
\end{equation*}
$$

where $S_{n}(\rho)=U_{n}(\rho / 2)$ are the Chebyshev polynomials of the second kind (A049310).
5. Difference of Chebyshev polynomials: if $\rho \neq 2$,

$$
\begin{equation*}
a_{n}=\frac{1}{\rho-2}\left(S_{n+1}(\rho)-S_{n}(\rho)-1\right), \quad n \in \mathbb{Z} \tag{6}
\end{equation*}
$$

For $n<0$, use $S_{-1}(\rho)=0$ and $S_{-n}(\rho)=-S_{n-2}(\rho)$ for $n \geq 2$.
6. Diophantine equation: If $x=a_{n}$ and $y=a_{n+1}$ are successive terms in the sequence, $n \in \mathbb{Z}$, then

$$
\begin{equation*}
x^{2}-x+y^{2}-y-\rho x y=0 \tag{7}
\end{equation*}
$$

These are all the integer solutions of the equation except when $\rho=3$ or $\rho=4$, for each of which an additional sequence of solutions exists, given in Appendix A.
7. Related binary quadratic form: define the sequence $b_{n}=(\rho-2) a_{n}+1$. If $u=b_{n}$ and $v=b_{n+1}$ are successive terms in the sequence, $n \in \mathbb{Z}$, then

$$
\begin{equation*}
u^{2}+v^{2}-\rho u v=-(\rho-2) \tag{8}
\end{equation*}
$$

8. Triangular number identity: if $\rho \geq 2$, then if $x=a_{n}$ and $y=a_{n+1}$ are successive terms in the sequence, $n \geq 0$,

$$
\begin{equation*}
\frac{T(x-1)+T(y-1)}{T(x+y-1)}=\frac{\rho}{\rho+2} \tag{9}
\end{equation*}
$$

where $T(i)$ is the $i$-th triangular number.
9. One-term formula giving squares:

$$
\begin{equation*}
\left(\rho^{2}-4\right) a_{n}^{2}+2(\rho+2) a_{n}+1=\left(a_{n+1}-a_{n-1}\right)^{2}, \quad n \in \mathbb{Z} \tag{10}
\end{equation*}
$$

10. Two-term formula giving squares:

$$
\begin{equation*}
4 a_{n+1} a_{n-1}+1=\left(2 a_{n}-1\right)^{2}, \quad n \in \mathbb{Z} \tag{11}
\end{equation*}
$$

11. Closed-form formulas: these are simplest if $n=0$ yields a value of 0 . Rather than change the offset, we give formulas for $a_{n-1}$.
(a) If $\rho \neq \pm 2$ :

$$
\begin{equation*}
a_{n-1}=\frac{1}{2-\rho}+c_{1} r_{1}^{n}+c_{2} r_{2}^{n}, \quad n \in \mathbb{Z} \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& r_{1}=\frac{1}{2}\left(\rho-\sqrt{\rho^{2}-4}\right) \\
& r_{2}=\frac{1}{2}\left(\rho+\sqrt{\rho^{2}-4}\right) \tag{13}
\end{align*}
$$

and

$$
\begin{align*}
& c_{1}=\frac{1}{2}\left(\frac{1}{\rho-2}-\frac{1}{\sqrt{\rho^{2}-4}}\right),  \tag{14}\\
& c_{2}=\frac{1}{2}\left(\frac{1}{\rho-2}+\frac{1}{\sqrt{\rho^{2}-4}}\right) .
\end{align*}
$$

(b) If $\rho=2$ :

$$
\begin{equation*}
a_{n-1}=\frac{n(n+1)}{2}, \quad n \in \mathbb{Z} \tag{15}
\end{equation*}
$$

(c) If $\rho=-2$ :

$$
\begin{equation*}
a_{n-1}=\frac{1}{4}\left(1-(-1)^{n}(2 n+1)\right), \quad n \in \mathbb{Z} \tag{16}
\end{equation*}
$$

We now give the proofs of the above statements.

## 3 Non-homogeneous linear recurrence

The proof of (2) proceeds by showing that the sequence the recurrence generates has the ordinary generating function (1). By definition,

$$
\alpha(x)=\sum_{n \geq 0} a_{n} x^{n}
$$

Using the recurrence (2),

$$
\begin{aligned}
\sum_{n \geq 0} a_{n+1} x^{n+1} & =\sum_{n \geq 0}\left(\rho a_{n}-a_{n-1}+1\right) x^{n+1} \\
\alpha(x)-a_{0} & =\rho x \sum_{n \geq 0} a_{n} x^{n}-a_{-1} x-x^{2} \sum_{n \geq 0} a_{n} x^{n}+x \sum_{n \geq 0} x^{n} \\
& =\rho x \alpha(x)-a_{-1} x-x^{2} \alpha(x)+\frac{x}{1-x} .
\end{aligned}
$$

Setting $a_{-1}=0$ and $a_{0}=1$ and solving for $\alpha(x)$ gives (1).

The recurrence (2) allows the sequence to be extended to negative $n$ by using it to solve for $a_{n-2}$ from $a_{n-1}$ and $a_{n}$. This yields $a_{-2}=0$ and $a_{-n}=a_{n-3}$ for $n>0$.

Observe that $a_{1}=\rho+1$. It is interesting that $a_{2}=\rho(\rho+1)$, which is twice a triangular number.

## 4 Homogeneous linear recurrence

Formula (3) follows at once by subtracting (2) from itself offset by 1 step.

## 5 Nonlinear 3-term identity

The identity (4) can be shown to hold by induction. The triple $\left(a_{-1}, a_{0}, a_{1}\right)=$ $(0,1, \rho+1)$ satisfies (4). Now assume

$$
a_{n+1} a_{n-1}-a_{n}\left(a_{n}-1\right)=0
$$

for some $n \geq 0$. Using (2),

$$
\begin{aligned}
a_{n+2} a_{n}-a_{n+1}\left(a_{n+1}-1\right) & =\left(\rho a_{n+1}-a_{n}+1\right) a_{n}-a_{n+1}\left(\rho a_{n}-a_{n-1}+1-1\right) \\
& =\rho\left(a_{n+1} a_{n}-a_{n+1} a_{n}\right)-a_{n}\left(a_{n}-1\right)+a_{n+1} a_{n-1} \\
& =0 .
\end{aligned}
$$

## 6 Partial sum of Chebyshev polynomials

The ordinary generating function of the Chebyshev polynomials is [4]

$$
\begin{equation*}
\sum_{n=0}^{\infty} S_{n}(\rho) x^{n}=\frac{1}{1-\rho x+x^{2}} \tag{17}
\end{equation*}
$$

Also, if $\alpha(x)$ is the generating function for a sequence $\left[a_{n}\right]$, then

$$
\frac{1}{1-x} \alpha(x)
$$

is the generating function for

$$
b_{n}=\sum_{i=0}^{n} a_{i} .
$$

Therefore (5) follows immediately from (1).

## 7 Difference of Chebyshev polynomials

We prove (6) by induction, using the following recurrence for the Chebyshev polynomials [4]:

$$
\begin{align*}
& S_{0}(\rho)=1 \\
& S_{1}(\rho)=\rho  \tag{18}\\
& S_{n}(\rho)=\rho S_{n-1}(\rho)-S_{n-2}(\rho), \quad n>0
\end{align*}
$$

Both $a_{-1}=0$ and $a_{0}=1$ satisfy (6), using $S_{-1}(\rho)=0$. Now assume that (6) holds up to some $n>0$. Then using (2),

$$
\begin{aligned}
a_{n+1} & =\rho a_{n}-a_{n-1}+1 \\
& =\rho \frac{S_{n+1}(\rho)-S_{n}(\rho)-1}{\rho-2}-\frac{S_{n}(\rho)-S_{n-1}(\rho)-1}{\rho-2}+1 \\
& =\frac{\left(\rho S_{n+1}(\rho)-S_{n}(\rho)\right)-\left(\rho S_{n}(\rho)-S_{n-1}(\rho)\right)-(\rho-1)+(\rho-2)}{\rho-2} \\
& =\frac{1}{\rho-2}\left(S_{n+2}(\rho)-S_{n+1}(\rho)-1\right)
\end{aligned}
$$

## 8 Diophantine equation

We prove (7) by induction. For $n=-1, x=0$ and $y=1$. These satisfy (7). Now for the inductive step, assume (rewriting (7) in a more convenient form)

$$
a_{n}\left(a_{n}-1\right)+a_{n+1}\left(a_{n+1}-1\right)-\rho a_{n} a_{n+1}=0
$$

for some $n \geq-1$. Then using (2),

$$
\begin{aligned}
a_{n+2}\left(a_{n+2}-1\right) & =\left(\rho a_{n+1}-a_{n}+1\right)\left(\rho a_{n+1}-a_{n}\right) \\
& =\rho^{2} a_{n+1}^{2}-2 \rho a_{n+1} a_{n}+\rho a_{n+1}+a_{n}\left(a_{n}-1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho a_{n+1} a_{n+2} & =\rho a_{n+1}\left(\rho a_{n+1}-a_{n}+1\right) \\
& =\rho^{2} a_{n+1}^{2}-\rho a_{n+1} a_{n}+\rho a_{n+1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_{n+1}\left(a_{n+1}-1\right)+a_{n+2}\left(a_{n+2}-1\right)-\rho a_{n+1} a_{n+2} & = \\
a_{n}\left(a_{n}-1\right)+a_{n+1}\left(a_{n+1}-1\right)-\rho a_{n} a_{n+1} & =0 .
\end{aligned}
$$

### 8.1 Completeness

The sequence $\left[a_{n}\right]$ given by (2) essentially provides all the solutions of (7), once a few details are accounted for. The proof and details are deferred to Appendix A.

### 8.2 Connection to a probability inversion problem

Observe that (7) rearranges as

$$
\begin{equation*}
\frac{2}{\rho+2}=\frac{2 x y}{(x+y)(x+y-1)}, \tag{19}
\end{equation*}
$$

provided $\rho \neq-2$ and $x+y \neq 0$ or 1. (We retain the 2 on both sides for the sake of the following interpretation.) If $\rho \geq 0$, then $0 \leq 2 /(\rho+2) \leq 1$. For solutions having $x \geq 0, y \geq 0$, and $x+y \geq 2$, (19) can be interpreted as the probability of drawing two different-colored socks at random from a drawer containing $x$ socks of one color and $y$ socks of another color. Hence the solutions of the Diophantine equation (7) solve the problem of finding numbers $x$ and $y$ that yield the probability given by a specific choice of $\rho$.

Of course, if $\rho$ is integer, not all ratios corresponding to arbitrary choices of $x$ and $y$ in (19) are represented. Allowing $\rho$ to be fractional, any rational probability can be represented. The problem of finding $x$ and $y$ that satisfy (19) in those cases is more complicated [3].

## 9 Related binary quadratic form

Setting $u=(\rho-2) x+1, v=(\rho-2) y+1$, where $(x, y)$ is a solution of $(7)$,

$$
\begin{aligned}
u^{2}+v^{2}-\rho u v= & ((\rho-2) x+1)^{2}+((\rho-2) y+1)^{2} \\
& -\rho((\rho-2) x+1)((\rho-2) y+1) \\
= & (\rho-2)\left((\rho-2) x^{2}+2 x+(\rho-2) y^{2}+2 y\right. \\
& =(\rho((\rho-2) x y+x+y))+2-\rho \\
= & -(\rho-2)
\end{aligned}
$$

This proves that $(u, v)$ satisfies (8).
These solutions form a family of integer sequences

$$
\begin{equation*}
b_{n}=(\rho-2) a_{n}+1 \tag{20}
\end{equation*}
$$

where, as for $a_{n}$, we write $b_{n}$ for $b_{n}(\rho)$. If $\rho=2, b_{n}=1, \forall n$, which satisfy (8) trivially. In this case, (20) is not invertible, and other solutions of (8) exist, but nonetheless the sequence $\left[b_{n}\right]$ is defined.

The members of this family of sequences obey the recurrence

$$
\begin{align*}
b_{-1} & =1 \\
b_{0} & =\rho-1  \tag{21}\\
b_{n} & =\rho b_{n-1}-b_{n-2}, \quad n>0
\end{align*}
$$

These sequences satisfy

$$
\begin{equation*}
b_{n}=S_{n+1}(\rho)-S_{n}(\rho), \tag{22}
\end{equation*}
$$

with $S_{n}(\rho)$ as defined in (18). Appendix B lists OEIS sequence A-numbers for some members of this family.

## 10 Triangular number identity

Subtracting each side of (19) from 1, we obtain (9):

$$
\frac{\rho}{\rho+2}=\frac{x(x-1)+y(y-1)}{(x+y)(x+y-1)}=\frac{T(x-1)+T(y-1)}{T(x+y-1)}
$$

where $T(n)=n(n+1) / 2$ is the $n$-th triangular number.

## 11 One-term formula giving squares

Using (2),

$$
\begin{aligned}
\left(a_{n+1}-a_{n-1}\right)^{2}= & \left(\rho a_{n}-2 a_{n-1}+1\right)^{2} \\
= & \rho^{2} a_{n}^{2}+4 a_{n-1}^{2}-4 \rho a_{n} a_{n-1}+2 \rho a_{n}-4 a_{n-1}+1 \\
= & \left(\rho^{2}-4\right) a_{n}^{2}+2(\rho+2) a_{n}+1+ \\
& 4\left(a_{n}^{2}-a_{n}+a_{n-1}^{2}-\rho a_{n} a_{n-1}-a_{n-1}\right)
\end{aligned}
$$

The final term in parentheses is 0 using the fact that $\left(a_{n-1}, a_{n}\right)$ is a solution of (7). This proves (10).

The sequence of terms that are squared is given by

$$
\begin{align*}
a_{n+1}(\rho)-a_{n-1}(\rho) & =S_{n+1}(\rho)+S_{n}(\rho)  \tag{23}\\
& =S_{2 n+2}(\sqrt{\rho+2})  \tag{24}\\
& =(-1)^{n+1} b_{n}(-\rho), \tag{25}
\end{align*}
$$

with $S_{n}(\rho)$ as in (18) and $b_{n}(\rho)$ as in (20). (Proofs omitted.) Appendix B lists OEIS sequence A-numbers for some of the sequences in this family.

## 12 Two-term formula giving squares

Using (4),

$$
\begin{aligned}
4 a_{n+1} a_{n-1}+1 & =4 a_{n}\left(a_{n}-1\right)+1 \\
& =\left(2 a_{n}-1\right)^{2}
\end{aligned}
$$

This proves (11). The sequence of terms that are squared is given by $2 a_{n}(\rho)-1$. Appendix B lists OEIS sequence A-numbers for some of the sequences in this family.

## 13 Closed-form formulas

We derive the closed-form formulas from the non-homogeneous recurrence formula (2). The associated homogeneous recurrence relation of (2) is

$$
a_{n}-\rho a_{n-1}+a_{n-2}=0
$$

whose characteristic equation is

$$
\begin{equation*}
r^{2}-\rho r+1=0 \tag{26}
\end{equation*}
$$

The roots of this equation are as in (13). They are distinct if $\rho \neq \pm 2$. Treating each of the cases in turn:

1. If $\rho \neq \pm 2$, the particular solution of (2) is of the form of a constant, $a_{n}=c_{0}, \forall n$. Inserting this into (2) and solving yields

$$
c_{0}=\frac{1}{2-\rho} .
$$

Combining the particular and the homogeneous solutions, the closed-form solution is of the form

$$
\begin{equation*}
c_{0}+c_{1} r_{1}^{n}+c_{2} r_{2}^{n}, \tag{27}
\end{equation*}
$$

where the constants $c_{1}$ and $c_{2}$ are determined by the initial conditions. Equating (27) to 0 for $n=0$ and 1 for $n=1$, we obtain (14), proving (12).
2. If $\rho=2$, the formula (15) satisfies (2) and agrees with the initial value $a_{0}=1$.
3. If $\rho=-2$, the particular solution of (2) is of the form of a constant, $a_{n}=c_{0}, \forall n$. Inserting this into (2) with $\rho=-2$ yields $c_{0}=1 / 4$. The roots (13) are $r_{1}=r_{2}=\rho / 2=-1$. Combining the particular and homogeneous solutions, the closed-form solution formula is of the form

$$
\begin{equation*}
c_{0}+(-1)^{n} c_{1}+(-1)^{n} n c_{2} . \tag{28}
\end{equation*}
$$

Equating (28) to 0 for $n=0$ and 1 for $n=1$ yields $c_{1}=-1 / 4, c_{2}=-1 / 2$, which gives (16).

## 14 Notable sequences

The following notable sequences are documented on OEIS at their respective A-numbers.

- For $\rho=-2\left(a_{n}=\right.$ A001057 $\left.(n+1)\right)$, the sequence including 0 is the canonical enumeration of the integers, interleaving the positive and negative integers.
- For $\rho=2\left(a_{n}=\underline{\text { A000217 }}(n+1)\right)$, the sequence is the triangular numbers.
- For $\rho=3\left(a_{n}=\right.$ A027941 $\left.(n+1)\right)$, the related sequence defined in (20) is $b_{n}=a_{n}+1=\underline{\text { A001519 }}(n+2)$, a bisection of the Fibonacci numbers: $b_{n}=F_{2 n+3}$ with $F_{n}=\underline{A 00045}(n)$ the Fibonacci numbers. The values $b_{n}$ are a subset of the Markov numbers A002559: $\left(1, b_{n}, b_{n+1}\right)$ is a Markov triple.
- For $\rho=14\left(a_{n}=A 076139(n+1)\right)$, the sequence consists of triangular numbers that are one-third of another triangular number: $a_{n}=T_{m}$ such that $3 T_{m}=T_{k}$ for some $k$, with $T_{n}=n(n+1) / 2$ the triangular numbers, $T_{n}=$ A000217 $(n)$.


## 15 Acknowledgements

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## Appendix A Completeness of solutions of (7)

We show that for any integer $\rho$, the sequence of values $\left[a_{n}\right]$ generated by the recurrence (2) provides the complete set of solutions of (7), except for two values of $\rho$ for which additional solutions exist that are generated using the same recurrence formula with different starting values.

In order for the sequence of solutions to be complete, elements for $n<0$ need to be included, in order to include the solutions $(0,0)$ and $(0,1)$, as well as those for which $x>y$, which by symmetry are also solutions.

To show that no solutions are missed, we need to recast (7) in a form for which the complete set of solutions can be readily found. Let

$$
\begin{equation*}
t=y+x, \quad s=y-x \tag{29}
\end{equation*}
$$

Then (7) becomes

$$
\begin{equation*}
(\rho-2) t^{2}+4 t-(\rho+2) s^{2}=0 \tag{30}
\end{equation*}
$$

If $\rho \neq 2$, we can complete the square on $t$ in (30). Setting

$$
\begin{equation*}
r=(\rho-2) t+2, \quad D=\rho^{2}-4 \tag{31}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
r^{2}-D s^{2}=4 \tag{32}
\end{equation*}
$$

(If $\rho$ is even, this equation can be divided through by 4 to obtain the Pell equation. For the sake of uniformity in the treatment, we leave it in this form.)

We note in passing that the Chebyshev polynomials satisfy the Pell equation

$$
T_{n}(x)^{2}-\left(x^{2}-1\right) U_{n-1}(x)^{2}=1,
$$

where $T_{n}(x)$ and $U_{n}(x)$ are the Chebyshev polynomials of the first and second kind, respectively [1]. Therefore

$$
\begin{equation*}
r_{n}=R_{n}(\rho), \quad s_{n}=S_{n-1}(\rho), \quad n \in \mathbb{Z} \tag{33}
\end{equation*}
$$

are solutions of $(32)$, where $R_{n}(\rho)=2 T_{n}(\rho / 2)$ and $S_{n}(\rho)=U_{n}(\rho / 2)$. The OEIS A-numbers for the sequences $\left[r_{n}\right]$ and $\left[s_{n}\right]$ for some values of $\rho$ are given in Appendix B.

We now treat the different cases for different values of $\rho$ in turn.

## A. 1 The case with $\rho=2$

If $\rho=2$, (30) reduces to the parabola $t=s^{2}$. Solving

$$
\begin{aligned}
& y+x=s^{2} \\
& y-x=s
\end{aligned}
$$

we obtain $x=s(s-1) / 2$ and $y=s(s+1) / 2$. Identifying $s$ with $n$, this result is the same as the closed-form solution (15). Hence for $\rho=2$ the sequence is the complete set of solutions.

## A. 2 The cases with $\rho=0, \pm 1$

If $|\rho|<2, D<0$ and (7) is a small ellipse, whose integer solution points can be found by inspection. The sequence $\left[a_{n}\right]$ repeats.

- $\rho=1$ : the sequence is the period- 6 repeat $[0,1,2,2,1,0, \ldots]$, generating the six solution points $(0,1),(1,2),(2,2),(2,1),(1,0),(0,0)$.
- $\rho=0$ : the sequence is the period-4 repeat $[0,1,1,0, \ldots]$, generating the four solution points $(0,1),(1,1),(1,0),(0,0)$.
- $\rho=-1$ : the sequence is the period-3 repeat $[0,1,0, \ldots]$, generating the three solution points $(0,1),(1,0),(0,0)$.


## A. 3 The case with $\rho=-2$

If $\rho=-2$, (30) reduces to $t(t-1)=0$, i.e., $y=-x$ or $y=1-x$. In this case, $(7)$ is a pair of straight lines of slope -1 , passing through $(0,0)$ and $(0,1)$ respectively. The recurrence $(2)$ with $\rho=-2$ takes the point $(n,-n)$ to $(-n, n+1)$ and $(-n, n+1)$ to $(n+1,-(n+1))$. Therefore, starting with the solution $(0,0)$ it generates half of the points on these two lines. Running the recurrence in reverse generates the rest, those of the form $(-n, n)$ and $(n+1,-n)$.

## A. 4 The cases with $\rho>2$ or $\rho<-2$

If $|\rho|>2, D>0$ and (7) is a hyperbola. For these cases, $D=\rho^{2}-4$ cannot be square. Therefore all solutions of (32) are given by

$$
\begin{equation*}
\frac{r+s \sqrt{D}}{2}= \pm\left(\frac{r_{1}+s_{1} \sqrt{D}}{2}\right)^{n}, \quad n=0, \pm 1, \pm 2, \ldots \tag{34}
\end{equation*}
$$

where $\left(r_{1}, s_{1}\right)$ is the solution with $r_{1}>0, s_{1}>0$ for which $r_{1}+s_{1} \sqrt{D}$ is least [2, Theorem 4.4]. Clearly $r_{1}=|\rho|$ and $s_{1}=1$. We can drop the absolute value since for $\rho<0$ it only changes the sign of $r$ and $s$. Inserting $\left(r_{1}, s_{1}\right)=(\rho, 1)$ into (34) and using the positive sign in front, for $n \geq 0$, we can write $(r, s)$ on the left in (34) as $\left(r_{n}, s_{n}\right)$ and obtain a recurrence formula:

$$
\begin{align*}
& r_{n}=\frac{1}{2}\left(\rho r_{n-1}+D s_{n-1}\right) \\
& s_{n}=\frac{1}{2}\left(r_{n-1}+\rho s_{n-1}\right) \tag{35}
\end{align*}
$$

Converting this to a recurrence formula for $x$ and $y$ using (31),

$$
\begin{aligned}
(\rho-2)\left(y_{n}+x_{n}\right)+2= & \frac{1}{2}\left(\rho\left((\rho-2)\left(y_{n-1}+x_{n-1}\right)+2\right)\right. \\
& \left.+\left(\rho^{2}-4\right)\left(y_{n-1}-x_{n-1}\right)\right) \\
y_{n}-x_{n}= & \frac{1}{2}\left((\rho-2)\left(y_{n-1}+x_{n-1}\right)+\rho\left(y_{n-1}-s_{n-1}\right)\right)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
(\rho-2)\left(y_{n}+x_{n}\right) & =(\rho-2)\left(1+(\rho+1) y_{n-1}-x_{n-1}\right), \\
y_{n}-x_{n} & =(\rho-1) y_{n-1}-x_{n-1}+1 .
\end{aligned}
$$

Cancelling $\rho-2$, which is nonzero, and solving, we obtain

$$
\begin{aligned}
& x_{n}=y_{n-1} \\
& y_{n}=\rho y_{n-1}-x_{n-1}+1
\end{aligned}
$$

which is the recurrence formula (2) with $x_{n}=a_{n-1}$ and $y_{n}=a_{n}$.
The solution $\left(r_{0}, s_{0}\right)=(2,0)$ is obtained with $n=0$ in (34). It maps to $(x, y)=(0,0)$. Therefore, using the recurrence (35) with $\left(r_{0}, s_{0}\right)=(2,0)$ gives a sequence of $\left(r_{n}, s_{n}\right)$ values that map to the same sequence of $a_{n}$ values as using (2) starting from $\left(a_{-2}, a_{-1}\right)=(0,0)$. Negative values of $s$ correspond to negative values of $n$ in (34), and simply interchange $x$ and $y$. These solutions are given by running (2) in reverse.

Finally, we need to consider values of $r$ with the opposite sign, corresponding to using the negative sign in front of the right-side expression in (34). For $\rho>2$ these map to the branch of the hyperbola in the third quadrant of $(x, y)$-space. The solution $\left(r_{0}, s_{0}\right)=(-2,0)$ maps to the vertex $(x, y)=\left(\frac{-2}{\rho-2}, \frac{-2}{\rho-2}\right)$. This is an integer solution only for $\rho \in\{0,1,3,4\}$. The values $\rho=0$ and 1 are in the elliptical regime treated earlier. For the values $\rho=3$ and 4 the vertex solutions
are $(x, y)=(-2,-2)$ and $(-1,-1)$ respectively. Using alternate starting values $a_{-2}=a_{-1}=-2$ or $a_{-2}=a_{-1}=-1$ in the recurrence (2) yields the same sequences of solutions as using (35) with $\left(r_{0}, s_{0}\right)=(-2,0)$ (equivalent to (34) with $n \geq 0$ and the minus sign in front). These negative sequences do not match OEIS sequences, but changing the sign they do. For $\rho=3$, the alternate $a_{n}=-$ A032908 $\left.n+2\right)$ and for $\rho=4$ it is $a_{n}=-$ A101879 $\left.n+2\right)$. Comments on these sequences by Robert Israel, Aug 262015 and Andrey Vyshnevyy, Sep 18 2015, respectively, note that consecutive terms of these sequences provide all positive integer pairs for which $K=(a+1) / b+(b+1) / a$ is integer. This is equivalent to (7) with $x=-a, y=-b$, and $\rho=K$.

If $\rho<-2$, the vertex of the hyperbola in $(x, y)$-space that is not coincident with the origin is never on integer coordinates. (For these values of $\rho$, the two branches of the hyperbola lie in the second and fourth quadrants, except for a short piece of one branch in the first quadrant between $(0,1)$ and $(1,0)$, with the other branch passing through $(0,0)$.) Therefore the only integer solutions of (7) for $\rho<-2$ are those given by the plus sign in front of the right-side expression in (34), equivalent to the recurrence (2) started with $(0,0)$.

This concludes the proof of completeness.

## Appendix B Related sequences

Here we list OEIS A-numbers for the sequences $\left[a_{n}\right]$ defined by (1), as well as for some related sequences. They are parameterized by $\rho$. The sequences $\left[b_{n}\right]$ given by (20) give solutions of the related binary quadratic form (8). The sequences $\left[r_{n}\right]$ and $\left[s_{n}\right]$ are the non-negative solutions of (32), given by (33).

The OEIS A-numbers for these sequences are listed in Table 1. The table includes $\rho=-1$ to 22 , which are the values of $\rho$ for the sequences listed at A212336 and that sequence itself.

The offsets between the sequences as defined here with $a_{0}=1$ and the OEIS sequences are shown: e.g., for $\rho=2$, the OEIS sequence for $a_{n}$ starts with 0 .

Table 2 gives OEIS A-numbers for the sequences of values whose squares are produced by the one-term formula (10) and the two-term formula (11). The index of the former is offset to match the offsets of the OEIS sequences.

## References

[1] Demeyer, Jeroen (2007). Diophantine Sets over Polynomial Rings and Hilbert's Tenth Problem for Function Fields. PhD Thesis, Universiteit Gent. Archived copy available.
[2] Hua, Luogeng (Loo Keng) (1982). Introduction to Number Theory, translated from the Chinese by Peter Shiu, Springer-Verlag, Berlin; Heidelberg; New York. Chapter 11.

Table 1: OEIS A-numbers for the sequences $\left[a_{n}\right]$ having the ordinary generating function (1), for $\left[b_{n}\right]$ as defined by (20), and for $\left[r_{n}\right]$ and $\left[s_{n}\right]$ as defined by (33).

| $\rho$ | $a_{n}$ | $b_{n}$ | $r_{n}$ | $s_{n}$ |
| :---: | :---: | :---: | :---: | :---: |
| -1 | A079978 $n$ ) | A131713 $n+1)$ | - | A102283 |
| 0 | A133872 ( $n$ ) | A087960 $(n+1)$ | - | A056594 $(n-1)$ |
| 1 | A164965 ( $n$ ) | A010892 $(n+2)$ | A057079 $(n+1)$ | A010892 $(n+2)$ |
| 2 | A000217 $(n+1)$ | A000012 $(n)$ | A007395 ( $n$ ) | A001477 ( $n$ ) |
| 3 | A027941 $(n+1)^{1}$ | A001519 $(n+2)$ | А005248 ( $n$ ) | А001906 $(n)$ |
| 4 | A061278 $(n+1)^{1}$ | A079935 $(n+1)$ | А003500 ( $n$ ) | А001353 $(n)$ |
| 5 | A089817 ( $n$ ) | A004253 $(n+1)$ | A003501 ( $n$ ) | А004254 ( $n$ ) |
| 6 | A053142 $(n+1)$ | A001653 $(n+1)$ | А003499 ( $n$ ) | А001109 $(n)$ |
| 7 | $\underline{\text { A092521 }}$ ( $n$ ) | A049685 $(n+1)$ | A056854 ( $n$ ) | A004187 ( $n$ ) |
| 8 | A076765 ( $n$ ) | A070997 $(n+1)$ | A086903 $(n)$ | А001090 $(n)$ |
| 9 | $\underline{\text { A092420 }}$ ( $n$ ) | A070998 $(n+1)$ | A056918 ( $n$ ) | A018913 ( $n$ ) |
| 10 | A097784 ( $n$ ) | A072256 $(n+2)$ | A087799 ( $n$ ) | A004189 ( $n$ ) |
| 11 | $\underline{\text { A097826 }}$ ( $n$ ) | A078922 $(n+1)$ | A057076 ${ }^{(n)}$ | A004190 $(n-1)$ |
| 12 | $\underline{\text { A097828 }}$ ( $n$ ) | A077417 $(n+1)$ | $\underline{\text { A087800 }}$ ( $n$ ) | A004191 $(n-1)$ |
| 13 | $\underline{\text { A097828 }}$ ( $n$ ) | A085260 $(n+1)^{2}$ | A078363 $(n)$ | A078362 $(n-1)$ |
| 14 | А076139 $(n+1)$ | A001570 $(n+1)$ | $\underline{\text { A067902 }}$ ( $n$ ) | A007655 ( $n$ ) |
| 15 | $\underline{\text { A097829 }}$ ( $n$ ) | A160682 $(n+1)$ | A078365 ( $n$ ) | A078364 $(n-1)$ |
| 16 | $\underline{\text { A097830 }}$ ( $n$ ) | A157456 $(n+1)$ | A090727 ( $n$ ) | A077412 $(n-1)$ |
| 17 | A097831 $(n)$ | A161595 $(n+1)$ | $\underline{\text { A078367 }}$ ( $n$ ) | A078366 $(n-1)$ |
| 18 | A049664 $(n+1)$ | А007805 $(n+1)$ | A087215 $(n)$ | А049660 $(n)$ |
| 19 | A097832 $(n)$ |  | A078369 ${ }^{\text {A090728 }}$ ( | A078368 $(n-1)$ |
| 20 | $\underline{\text { A097833 }}$ ( $n$ ) | A075839 $(n+1)$ | $\underline{\overline{\text { A090728 }}}$ ( $n$ ) | $\underline{\text { A075843 }}$ ( $n$ ) |
| 21 | A212335 $(n)$ | , | $\underline{\text { A090729 }}$ ( $n$ ) | $\overline{\text { A092499 }}$ ( $n$ ) |
| 22 | A212336 $(n)$ | A157014 $(n+1)$ | А090730 $(n)$ | A077421 $(n-1)$ |

${ }^{1}$ As noted in Appendix A, $\rho=3$ and $\rho=4$ each have an additional sequence of integer solutions of (7) on the negative branch of the hyperbola. The alternate solutions are: A032908 $n+2$ ) for $\rho=3$ and and A101879 $n+2)$ for $\rho=4$.
${ }^{2}$ Based on sequence description "Ratio-determined insertion sequence $\mathrm{I}(0.0833344)$," identification of this sequence with $b_{n}(13)$ as defined here is conjectural. See comment by M. F. Hasler Nov 052018.

Table 2: OEIS A-numbers for the sequences $\left[a_{n}-a_{n-2}\right.$ ] and $\left[2 a_{n}-1\right]$ for $\rho=$ -1 to 22 . These sequences are the values whose squares are produced by the formulas (10) and (11) respectively.

| $\rho$ | $a_{n}-a_{n-2}$ | $2 a_{n}-1$ |
| :---: | :---: | :---: |
| -1 | A057078 $n$ ) | - |
| 0 | A057077 ( $n$ ) | A057077 ( $n$ ) |
| 1 | A057079 ( $n$ ) | - A130778 ${ }^{(n)}$ |
| 2 | A005408 $(n)$ | A028387 ( $n$ ) |
| 3 | A002878 $(n)$ | - |
| 4 | A001834 ( $n$ ) | - |
| 5 | A030221 ( $n$ ) | - |
| 6 | A002315 ( $n$ ) | - |
| 7 | A033890 $(n)$ | - |
| 8 | A057080 $(n)$ | - |
| 9 | A057081 ( $n$ ) | - |
| 10 | $\underline{\text { A054320 }}$ ( $n$ ) | - |
| 11 | $\underline{\text { A097783 }}$ ( $n$ ) | - |
| 12 | A077416 $(n)$ | - |
| 13 | A126866 ${ }^{(n)}$ | - |
| 14 | A028230 $(n)$ | - |
| 15 | A161591 $(n)$ | - |
| 16 | A159678 ${ }^{(n)}$ | - |
| 17 | A161599 $(n)$ | - |
| 18 | A049629 $(n)$ | - |
| 19 | - | - |
| 20 | A083043 ( $n$ ) | - |
| 21 | - | - |
| 22 | A133283 ( $n$ ) | - |

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