Proof of Conjecture 2 and Related Properties of A209260.

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Let $T(i,j) = \frac{j}{2} \times (2 \times i - j + 1)$, for all $i, j \in \mathbb{N}$ and $1 \le j \le i$, be the function defining the triangle in A141419.

For all $v \in \mathbb{N}$ define the set $B_v = \{ d + \frac{h-1}{2} \mid v = d \times h \ \& \ d, h \in \mathbb{N} \ \& \ h \text{ odd} \} \subset \mathbb{N}$.

Theorem: For all $v \in \mathbb{N}$, v is in row i of A141419 if and only if $i \in B_v$.

"⇒": Solving equation $v = T(i,j)$ with $1 \le j \le i$ gives $j = \frac{1}{2} + i - \sqrt{\left(\frac{1}{2} + i\right)^2 - 2 \times v}$ as the only possible solution. Since $j \in \mathbb{N}$, there is $k \in \mathbb{N}$, $0 \le k < i$, such that $\sqrt{\left(\frac{1}{2} + i\right)^2 - 2 \times v} = \frac{1}{2} + k$ which is equivalent to $(i - k) \times (i + k + 1) = 2 \times v$ and $i - k = j$.

If j is odd then $j \mid v$ and $\frac{i+k+1}{2} = \frac{v}{j}$, in other words $i = \frac{j-1}{2} + \frac{v}{j} \in B_v$.

If j is even then $i + k + 1$ is odd and $\frac{i-k}{2} \times (i + k + 1) = v$ is an integer factorization. Therefore, $\frac{i}{2} = \frac{k}{2} + \frac{v}{i+k+1}$ which leads to $i = \frac{(i+k+1)-1}{2} + \frac{v}{i+k+1} \in B_v$.

In either case, if $v = T(i,j)$ then $i \in B_v$.

"⇐": Let $i = d + \frac{h-1}{2} \in B_v$ with $v = d \times h$, $d,h \in \mathbb{N}$ and h odd. Then $T(d + \frac{h-1}{2}, z) = d \times h$ is equivalent to $(z - 2 \times d) \times (z - h) = 0$. Since $\frac{h+1}{2} > d$ is equivalent to $d + \frac{h-1}{2} < h$ and since $\frac{h+1}{2} \le d$ is equivalent to $d + \frac{h-1}{2} < d + \frac{h+1}{2} \le 2 \times d$ exactly one of the two solutions $z = 2 \times d$ and $z = h$ is in the index range $1 \ldots d + \frac{h-1}{2}$. Therefore, either $v = T(d + \frac{h-1}{2}, 2 \times d)$ or $= T(d + \frac{h-1}{2}, h)$.

Corollary 1: The row of triangle A209260 in which $v \in \mathbb{N}$ occurs is the least element of B_v.

Corollary 2: Let $v = d \times h$ with $d,h \in \mathbb{N}$ and h odd. Then $v = T(d + \frac{h-1}{2}, h)$ when $\frac{h+1}{2} \le d$ and $v = T(d + \frac{h-1}{2}, 2 \times d)$ when $\frac{h+1}{2} > d$.