On the determinant of a symmetric matrix

Sela Fried

This note is concerned with sequence A204238, which corresponds to the determinant of a certain symmetric matrix. We obtain a closed form formula for A204238(n), which immediately confirms conjectures by Barker.

Theorem 1. *Let $n \in \mathbb{N}$ and let M_n be the symmetric matrix of size n given by $(M_n)_{ij} = f(i,j) = \max(3i-j, 3j-i)$. Set $a(n) = \det(M_n)$. Then,*

$$a(n) = (-4)^{n-2}(1-9n). \tag{1}$$

Proof. Clearly, $f(i,j) = 3(i+j) - 4\min(i,j)$. Define matrices A_n and K_n by $(A_n)_{ij} = i+j$ and $(K_n)_{ij} = \min(i,j)$, respectively. Then $M_n = 3A_n - 4K_n$ and therefore $\det(M_n) = (-4)^n \det(K_n - \frac{3}{4}A_n)$. It is folklore that $\det(K_n) = 1$. One way to see this (see comment by Berry in A003983) is to write $K_n = L_n L_n^T$, where L_n is the square matrix of size n defined by

$$(L_n)_{ij} = \begin{cases} 1, & \text{if } i \geq j, \\ 0, & \text{if } i < j. \end{cases}$$

Then, $\det(K_n) = \det(L_n)^2 = 1$. Thus, K_n is invertible and by direct calculations one verifies that K_n^{-1} is the tridiagonal matrix given by

$$(K_n^{-1})_{ii} = \begin{cases} 2, & \text{if } 1 \leq i \leq n-1, \\ 1, & \text{if } i = n, \end{cases}$$
$$(K_n^{-1})_{i,i+1} = (K_n^{-1})_{i+1,i} = -1.$$

Let $\mathbf{1} = (1, \ldots, 1)^\mathsf{T}$ and $u = (1, 2, \ldots, n)^\mathsf{T}$. Then $-\frac{3}{4}A_n = UV^\mathsf{T}$ where U and V are the two matrices of size $n \times 2$ defined by $U = -\frac{3}{4}\begin{pmatrix} u & \mathbf{1} \end{pmatrix}$ and $V = \begin{pmatrix} \mathbf{1} & u \end{pmatrix}$, respectively. By the matrix determinant lemma,

$$\det(K_n + UV^\mathsf{T}) = \det(K_n)\det(I_2 + V^\mathsf{T}K_n^{-1}U) = \det(I_2 + V^\mathsf{T}K_n^{-1}U).$$

It is easily verified that

$$I_2 + V^\mathsf{T}K_n^{-1}U = I_2 - \frac{3}{4}\begin{pmatrix} 1 & 1 \\ n & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \\ -\frac{3n}{4} & \frac{1}{4} \end{pmatrix}.$$

Thus,

$$a(n) = \det(M_n) = (-4)^n \det(K_n - \tfrac{3}{4}A_n) = (-4)^n \det(K_n + UV^{\mathsf{T}}) = (-4)^{n-2}(1-9n).$$

$\square$

Corollary 1. *Let $n \in \mathbb{N}$ such that $n \geq 3$. Then*

$$a(n) = -8a(n-1) - 16a(n-2).$$

Furthermore, the ordinary generating function for the $a(n)$s is given by

$$-\frac{x(x-2)}{(4x+1)^2}.$$

Proof. It follows from the closed form formula for $a(n)$ that $a(n)$ is annihilated by $(E+4)^2$, where E is the shift operator $Ea(n) = a(n+1)$. Expanding $(E+4)^2 = E^2 + 8E + 16$, we obtain the asserted recurrence, from which, the ordinary generating function immediately follows. $\square$

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.