Maple-assisted proof of formula for A203455

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There are $3^6 = 729$ configurations for a 2×3 sub-array. Consider the 729×729 transition matrix T with entries $T_{ij} = 1$ if the bottom two rows of a 3×3 sub-array could be in configuration i while the top two rows are in configuration j (i.e. the top row of configuration i is the bottom row of configuration j, and every nonzero element of the middle row is less than or equal to at least two of its neighbours), and 0 otherwise. The following Maple code computes it. Configurations are encoded as 6-element lists, corresponding to base-3 digits, in the order

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

```
> Configs:=  [seq(convert(3^6+i,base,3)[1..6],i=0..3^6-1)]:
> Compatible:= proc(i,j)
   if not Configs[i][1..3]=Configs[j][4..6] then return 0 fi;
   if Configs[i][1] <= sort([Configs[i][2],Configs[i][4],Configs[j][1]])[2]
   and Configs[i][2] <= sort([Configs[i][1],Configs[i][3],Configs[j][2],Configs[i][5]])[3]
   and Configs[i][3] <= sort([Configs[i][2],Configs[i][6],Configs[j][3]])[2]
   then 1 else 0
   fi
  end proc:
> T:= Matrix(729,729,Compatible):
```

Thus for $n \geq 2$ $a(n) = u\,T^{n-2} v$ where u is a row vector with entries 1 for configurations where each bottom row entry is less than or equal to at least two neighbours, 0 otherwise, and v is a column vectors with entries 1 for configurations where each top row entry is less than or equal to at least two neighbours. The following Maple code produces these vectors.

```
> u:= Vector[row](729, i -> `if`(Configs[i][4] <= min(Configs[i][1],Configs[i][5]) and
  Configs[i][5] <= sort([Configs[i][4],Configs[i][2],Configs[i][6]])[2] and
  Configs[i][6] <= min(Configs[i][5],Configs[i][3]),1,0)):
  v:= Vector[column](729, i -> `if`(Configs[i][1] <= min(Configs[i][2],Configs[i][4]) and
  Configs[i][2] <= sort([Configs[i][1],Configs[i][5],Configs[i][3]])[2] and
  Configs[i][3] <= min(Configs[i][2],Configs[i][6]),1,0)):
```

To check, here are the first few entries of our sequence.

```
> TV[0]:= v:
  for n from 1 to 37 do TV[n]:= T . TV[n-1] od:
> A:= [seq(u . TV[n],n=0..37)];
```

$A :=$ [9, 84, 543, 3748, 29677, 237999, 1882149, 14936068, 119307906, 954802767, 7642229473, 61191946032, 490168235993, 3927147409735, 31465890290038, 252129334037581, 2020326252175096, 16189287109355756, 129729279431146577, **(1)** 1039562617181770195, 8330379061723434337, 66754371165261697369, 534927732148162788013, 4286578488372677625273, 34349988864193761288544, 275259622149862420081684, 2205761073316883716939176, 17675611825350092422659781, 141641481037984605565190944, 1135027731164936977224242486, 9095414401685995737629888306, 72885059505015825124501888502, 584056062328439328864801789649, 4680266258354642289286133135400, 37504776837770777459246625202919, 300540227645462881137963202899768, 2408344645363305295820801801289554, 19298993608102086490128889316759748]

Note that $a(1) = 1$ is not included here. The verification of the empirical recurrence for $n = 37$, which involves $a(1)$, must be done separately.

```
> n:= 'n':
  empirical:= a(n) = 18*a(n-1) -133*a(n-2) +640*a(n-3) -2309*a(n-4)
  +5762*a(n-5) -9918*a(n-6) +8864*a(n-7) +2970*a(n-8) -18151*a(n-9)
  +23183*a(n-10) +21799*a(n-11) -75109*a(n-12) +56329*a(n-13)
  -9892*a(n-14) -115814*a(n-15) +105665*a(n-16) +64162*a(n-17)
  -38348*a(n-18) +8767*a(n-19) +54790*a(n-20) -95246*a(n-21)
  -139155*a(n-22) +81259*a(n-23) +104482*a(n-24) +18968*a(n-25)
  -2909*a(n-26) -41630*a(n-27) -37578*a(n-28) +6670*a(n-29) +17693*
  a(n-30) +5629*a(n-31) -713*a(n-32) -928*a(n-33) -300*a(n-34) -52*
  a(n-35) -4*a(n-36):
> eval(empirical, {a(n-36)=1, seq(a(n-i)=A[36-i],i=0..35)});
```

$$300540227645462881137963202899768 = 300540227645462881137963202899768 \quad (2)$$

Now here is the minimal polynomial P of T, as computed by Maple.

```
> P:= unapply(LinearAlgebra:-MinimalPolynomial(T, t), t);
```

$$P := t \mapsto -28\,t^3 - 4\,t^2 - 76\,t^4 + t^{63} - 27\,t^{62} + 345\,t^{61} - 2906\,t^{60} + 18194\,t^{59} - 89630\,t^{58} + 358894\,t^{57} - 1185747\,t^{56} + 3260528\,t^{55} - 7466097\,t^{54} + 14160132\,t^{53} - 21896170\,t^{52} + 26633596\,t^{51} - 23529983\,t^{50} + 11654811\,t^{49} + 723584\,t^{48} + 1889472\,t^{47} - 30150715\,t^{46} + 76789441\,t^{45} - 108264425\,t^{44} + 86458925\,t^{43} + 2171553\,t^{42} - 134672418\,t^{41} + 234146243\,t^{40} - 241899114\,t^{39} + 161267275\,t^{38} - 17371932\,t^{37} - 86178714\,t^{36} + 121036721\,t^{35} - 159172848\,t^{34} + 142380806\,t^{33} - 98877622\,t^{32} + 56252050\,t^{31} + 58219355\,t^{30} - 89668896\,t^{29} + 59655755\,t^{28} - 88313022\,t^{27} + 12314620\,t^{26} + 52223803\,t^{25} + 6093896\,t^{24} + 8246178\,t^{23} - 26031408\,t^{22} - 11882831\,t^{21} - 3875046\,t^{20} + 10990860\,t^{19} + 10242618\,t^{18} - 3330927\,t^{17} - 2031952\,t^{16} - 1747833\,t^{15} - 1001573\,t^{14} + 1169345\,t^{13} + 559815\,t^{12} - 163580\,t^{11} - 94865\,t^{10} - 16502\,t^9 + 321\,t^8 + 4403\,t^7 + 1435\,t^6 + 16\,t^5 \quad (3)$$

```
> degree(P(t));
```

$$63 \quad (4)$$

This turns out to have degree 63, but with the t^0 and t^1 coefficients 0. Thus we will have

$$0 = u\,P(T)\,T^n\,v = \sum_{i=2}^{63} p_i\,b(i+n)$$ where p_i is the coefficient of t^i in $P(t)$. That corresponds to a homogeneous linear recurrence of order 61, which would hold true for any u and v, after a delay of 2. It seems that with our particular u and v we have a recurrence of order only 36, corresponding to a factor of P .

```
> Q:= unapply(add(coeff((lhs-rhs)(empirical),a(n-i))*t^(36-i),i=0..36),t);
```

$$Q := t \mapsto 4 + 52\,t + 928\,t^3 + 300\,t^2 + 713\,t^4 + t^{36} - 18\,t^{35} + 133\,t^{34} - 640\,t^{33} + 2309\,t^{32} - 5762\,t^{31} + 9918\,t^{30} - 8864\,t^{29} - 2970\,t^{28} + 18151\,t^{27} - 23183\,t^{26} - 21799\,t^{25} + 75109\,t^{24} - 56329\,t^{23} + 9892\,t^{22} + 115814\,t^{21} - 105665\,t^{20} - 64162\,t^{19} + 38348\,t^{18} - 8767\,t^{17} - 54790\,t^{16} + 95246\,t^{15} + 139155\,t^{14} - 81259\,t^{13} - 104482\,t^{12} - 18968\,t^{11} + 2909\,t^{10} + 41630\,t^9 + 37578\,t^8 - 6670\,t^7 - 17693\,t^6 - 5629\,t^5 \tag{5}$$

The complementary factor $R(t) = \dfrac{P(t)}{Q(t)}$ has degree 27, again with the lowest two coefficients 0.

```
> R:= unapply(normal(P(t)/Q(t)),t);
```

$$R := t \mapsto (-1 + 6\,t + 72\,t^3 - 22\,t^2 - 141\,t^4 + t^{25} - 9\,t^{24} + 50\,t^{23} - 169\,t^{22} + 433\,t^{21} - 816\,t^{20} + 1231\,t^{19} - 1494\,t^{18} + 1392\,t^{17} - 948\,t^{16} + 133\,t^{15} + 728\,t^{14} - 968\,t^{13} + 1031\,t^{12} - 921\,t^{11} + 372\,t^{10} - 101\,t^9 + 5\,t^8 + 190\,t^7 - 200\,t^6 + 161\,t^5)\,t^2 \tag{6}$$

```
> degree(R(t));
```

$$27 \tag{7}$$

Now we want to show that $c(n) = u\,Q(T)\,T^n\,v = 0$ for all $n \geq 0$. This will certainly satisfy the recurrence

$$\sum_{i=2}^{27} r_i\,c(i+n) = \sum_{i=2}^{27} r_i u\,Q(T)\,T^{n+i} v = u\,Q(T)\,R(T)\,T^n\,v = u\,P(T)\,T^n\,v = 0$$

where r_i are the coefficients of $R(t)$. To show all $c(n) = 0$ it suffices to show $c(1) = \ldots = c(26) = 0$.

First we compute $w = u\,Q(T)$, then multiply it with the already-computed $T^n v$.

```
> UT[0]:= u:
  for n from 1 to 36 do UT[n]:= UT[n-1].T od:
  w:= add(coeff(Q(t),t,j)*UT[j],j=0..degree(Q(t))):
> seq(w . TV[n],n=0..26);
```

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \tag{8}$$

This completes the proof.