On A203246

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Let $n \geq 2$ be an integer. The sequence $a(n)$ is defined to be the second elementary symmetric function of the first n terms of $(1, 1, 2, 2, 3, 3, 4, 4, \ldots)$. In what follows we obtain a closed-form formula for $a(n)$. It is slightly different from the one conjectured by Mathar. Nevertheless both formulas coincide. Thus, we confirm the conjectures stated there. The same method should apply to the analogue sequences A203298 and A203299.

$$
\begin{aligned}
a(n) &= \sum_{j=1}^{n-1} \left\lceil \frac{j}{2} \right\rceil \sum_{k=j+1}^{n} \left\lceil \frac{k}{2} \right\rceil \\
&= \frac{1}{2} \sum_{j=1}^{n-1} \left\lceil \frac{j}{2} \right\rceil \left(\left\lfloor \frac{(n+1)^2}{2} \right\rfloor - \left\lfloor \frac{(j+1)^2}{2} \right\rfloor \right) \\
&= \frac{1}{4} \left\lfloor \frac{(n+1)^2}{2} \right\rfloor \left\lfloor \frac{n^2}{2} \right\rfloor - \frac{1}{2} \sum_{j=1}^{n-1} \left\lceil \frac{j}{2} \right\rceil \left\lfloor \frac{(j+1)^2}{2} \right\rfloor \\
&= \begin{cases} \frac{n^2(n+1)^2 - n^2}{16} - \frac{1}{4} \sum_{j=1}^{\frac{n-2}{2}} \left(j(2j+1)^2 - j\right) - \sum_{j=1}^{\frac{n}{2}} j^3 & \text{if } n \text{ is even} \\ \frac{(n+1)^2(n^2-1)}{16} - \frac{1}{4} \sum_{j=1}^{\frac{n-1}{2}} \left(j(2j+1)^2 - j\right) - \sum_{j=1}^{\frac{n-1}{2}} j^3 & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{n^4}{32} + \frac{n^3-n}{12} & \text{if } n \text{ is even} \\ \frac{n^4}{32} + \frac{n^3-n}{12} + \frac{2n^2-3}{32} & \text{if } n \text{ is odd} \end{cases} \\
&= \frac{n^4}{32} + \frac{n^3 - n}{12} + \frac{2n^2 - 3}{64} + (-1)^{n+1} \frac{2n^2 - 3}{64}.
\end{aligned}
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