Proof of empirical formula for A202785

Robert Israel

May 2, 2019

Let C be the polytope in $\mathbb{R}^{3\times 3}$ defined by the equations $a_{i,1}+a_{i,2}+a_{i,3} = a_{1,1}+a_{1,2}+a_{1,3}$ for $i = 2, 3$ and $a_{1,j} + a_{2,j} + a_{3,j} = a_{1,1} + a_{1,2} + a_{1,3}$ for $j = 1, 2, 3$ and the inequalities $0 \le a_{i,j} \le 1$ for $i, j = 1, 2, 3$. Then $A202785(n) = L(C, n) = \#(nC \cap \mathbb{Z}^{3\times 3})$ is the number of integer lattice points in the dilation nC.

Note that there are really only four linearly independent constraints (since the sum of sums of the rows is equal to the sum of sums of the columns). Thus the dimension of the polytope is 5.

I claim that the extreme points of C all have integer coordinates. If this is the case, then Ehrhart[1] showed that $L(C, n)$ is a polynomial in n whose degree is the dimension of C, namely 5 in this case. The formula for the sequence can then be determined using the Data.

To prove the claim, note that any extreme point must have at least five of the variables with values 0 or 1. For each choice of five of the 9 variables and each choice of 0 or 1 for the values of those variables, we attempt to solve the system obtained by substituting those values. If this system has a unique solution and the solution is in the polytope, that solution is one of the extreme points. There are $\binom{9}{5} \times 2^5 = 4032$ cases to consider. The result is that the polytope has 14 extreme points, namely: all zeros, all ones, the six permutaion matrices, and the six matrices $J - P$ where J is the matrix of all ones and P a permutation matrix. All of these are in $\mathbb{Z}^{3\times 3}$.

References

[1] E. Ehrhart, *Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux*, J.Reine Angew. Math. **226** (1967), 1–29.