OEIS A197008

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ABSTRACT. The constants in sequences [1, A197138..A197155] are roots of cubic polynomials with coefficients defined by the (h, k) coordinates of the point in the first quadrant on one triangle side and by the slope m of the equation of the line for the third triangle side.

1. MINIMIZING DISTANCE FROM *x*-AXIS TO DIAGONAL y = mx

The equation of a bundle of lines with inclination α that run through the point (x, y) = (h, k) is

(1)
$$y = \alpha(x-h) + k.$$

These lines intersect the horizontal axis at

(2)
$$\alpha(x-h) + k = 0$$

which has the solution

(3)
$$(x_1, y_1) = (h - \frac{k}{\alpha}, 0).$$

These lines intersect the diagonal y = mx at

(4)
$$\alpha(x-h) + k = mx$$

which has the solution

(5)
$$(x_2, y_2) = \left(\frac{\alpha h - k}{\alpha - m}, m \frac{\alpha h - k}{\alpha - m}\right).$$

The squared Euclidean distance between the intersections of the horizontal line and the diagonal is

(6)
$$d^{2} = (x_{1} - x_{2})^{2} + (y_{1} - y_{2})^{2} = \frac{m^{2}(\alpha h - k)^{2}(1 + \alpha^{2})}{\alpha^{2}(\alpha - m)^{2}}$$

The Philo Line is defined by the minimum of that distance at negative α

Remark 1. The global minimum at inclination $\alpha = k/h$ does not define a triangle but means that the horizontal line, the diagonal and the line of the bundle all intersect at (0,0).

An arithmetic expression for the location of the minimum is obtained by setting the derivative $\partial d^2/\partial \alpha = 0$, so

(7)
$$-2m^2 \frac{(h\alpha - k)[(mh - k)\alpha^3 + h\alpha^2 - 2k\alpha + km]}{\alpha^3(\alpha - m)^3} = 0$$

equivalent to the root of the polynomial

(8)
$$(mh-k)\alpha^3 + h\alpha^2 - 2k\alpha + km = 0$$

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Inverting (3) as $\alpha_1 = k/(h - x_1)$ and plugging this into (8) one finds that x_1 is a root of the polynomial

(9)
$$mx^{3} + (2k - 3mh)x^{2} + 3h(mh - k)x - (mh - k)(h^{2} + k^{2}).$$

Plugging in the same expression in (6) gives

(10)
$$d^{2} = \frac{k^{2} + x^{2} - 2xh + h^{2}}{(k + mx - mh)^{2}}x^{2}m^{2}.$$

Remark 2. The numerator of the squared distance (6) is a quartic polynomial in α ,

(11)
$$m^2(\alpha h - k)^2(1 + \alpha^2)$$

which can be reduced with (8) to a quadratic polynomial

2. Minimizing distance from x-axis to y-axis

The equation of a bundle of lines with inclination α that run through the point (x, y) = (h, k), h, k > 0, has an intersection with the x-axis given by (3).

These lines intersect the y-axis at

(13)
$$\alpha(-h) + k$$

which has the solution

(14)
$$(x_3, y_3) = (0, k - \alpha h).$$

The squared Euclidean distance between the intersections of the horizontal line and vertical lines is

(15)
$$d^{2} = (x_{1} - x_{3})^{2} + (y_{1} - y_{3})^{2} = \frac{(\alpha h - k)^{2}(1 + \alpha^{2})}{\alpha^{2}}.$$

The Philo Line is defined by the minimum of that curve (at negative α).

An arithmetic expression for the location of the minimum is obtained by setting the derivative $\partial d^2/\partial \alpha = 0$, so

(16)
$$2\frac{(h\alpha - k)(h\alpha^3 + k)}{\alpha^3} = 0$$

equivalent to

(17)
$$\alpha = -\sqrt[3]{k/h}$$

and therefore in [1, A197008]

(18)
$$d = \frac{k - \alpha h}{|\alpha|} \sqrt{1 + \alpha^2} = h [1 + (k/h)^{2/3}]^{3/2}$$

Inverting (3) as $\alpha_1 = k/(h - x_1)$ and plugging this into (17) one finds

(19)
$$x_1 = h + k \sqrt[3]{k/h}.$$

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References

 [1] O. E. I. S. Foundation Inc., The On-Line Encyclopedia Of Integer Sequences, (2022), https://oeis.org/. MR 3822822
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