# Even outdegree rooted trees <br> <br> DAVID CALLAN 

 <br> <br> DAVID CALLAN}

## July 7, 2014

Let $b_{n}$ (resp. $c_{n}$ ) denote the number of rooted trees (unordered, unlabeled) with $n$ vertices in which all (resp. all non-root) vertices have even outdegree. Let $B(x)=\sum_{n \geq 1} b_{n} x^{n}$ and $C(x)=\sum_{n \geq 1} c_{n} x^{n}$ denote the respective generating functions.

First, note that if all vertices have even outdegree, the tree must have an even number of edges, hence an odd number of vertices. So $b_{n}=0$ for even $n$. Moreover, if all non-root vertices have even outdegree and the total number of vertices is odd (hence, even number of edges), then the root must also have even outdegree. So $b_{n}=c_{n}$ for $n$ odd. These observations imply

$$
\begin{equation*}
B(x)=\frac{C(x)-C(-x)}{2} \tag{1}
\end{equation*}
$$

Next, to obtain an expression for $c_{n}$ we use the standard method for rooted trees $[1$, Sec. 2.3.4.4]. For a tree on $n$ vertices, all of even outdegree, consider the subtrees of the root. Say $i_{1}$ have 1 vertex, $i_{2}$ have 2 vertices, and so on, with $i_{1}+2 i_{2}+\cdots+(n-1) i_{n-1}=n-1$ $\left(1^{i_{1}} 2^{i_{2}} \ldots(n-1)^{i_{n-1}}\right.$ defines a partition of $\left.n-1\right)$. All vertices in the subtrees have even outdegree, so there are

$$
\binom{b_{k}+i_{k}-1}{i_{k}}
$$

ways to choose the $i_{k}$ subtrees with $k$ vertices from the pool of $b_{k}$ such trees (since repetition is allowed), leading to

$$
\begin{equation*}
c_{n}=\sum_{i_{1}+2 i_{2}+\cdots=n-1}\binom{b_{1}+i_{1}-1}{i_{1}} \ldots \ldots\binom{b_{n-1}+i_{n-1}-1}{i_{n-1}} \quad \text { for } n>1 \tag{2}
\end{equation*}
$$

The expansion $\left(1-x^{k}\right)^{-b}=\sum_{i}\binom{b+i-1}{i} x^{i k}$, together with (2), implies

$$
\begin{equation*}
C(x)=x \prod_{k \geq 1}\left(1-x^{k}\right)^{-b_{k}} \tag{3}
\end{equation*}
$$

Now take logs and use the expansion $-\log (1-y)=y+y^{2} / 2+y^{3} / 3+\cdots$ to obtain

$$
\begin{equation*}
C(x)=x e^{\sum_{k \geq 1} \frac{B\left(x^{k}\right)}{k}} . \tag{4}
\end{equation*}
$$

Equivalently, using (1), we have shown that $C(x)$ satisfies the defining equation

$$
\begin{equation*}
C(x)=x e^{\sum_{k \geq 1} \frac{C\left(x^{k}\right)-C\left(-x^{k}\right)}{k}} \tag{5}
\end{equation*}
$$

Equation (5) implies that $A(x):=C(x) / x$ satisfies the title identity for A195865.

## References

[1] D. Knuth, The Art of Computer Programming, Sorting and Searching, Vol. 1, 3rd ed., Addison-Wesley Professional, Boston, MA, 1997.

