Circles Passing through 3 Distinct Points of the Square Lattice.

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The two sequences A192493 and A192494 give the squared radii of the circumcircles of non-degenerate triangles formed by 3 points of the square lattice, sorted by increasing size.

R^2(n) = A192493(n) / A192494(n).

To get minimal enclosing circles of m>1 given points of the square lattice, also circles supported only by two points forming the diameter of the enclosing circle would have to be considered.
However, there are only 5 squared radii of minimal enclosing circles not representable by circumcircles of non-degenerate triangles of lattice points:

R^2 = (1/4) * {1, 9, 49, 81, 121}, which are not included in the 2 sequences.

The representation starts to become non-unique for R^2 >= 9425 / 578. This is the first case, where lattice points supporting triangles with the same circumradius are found on circles with different centers, even when moved to a normalized location.

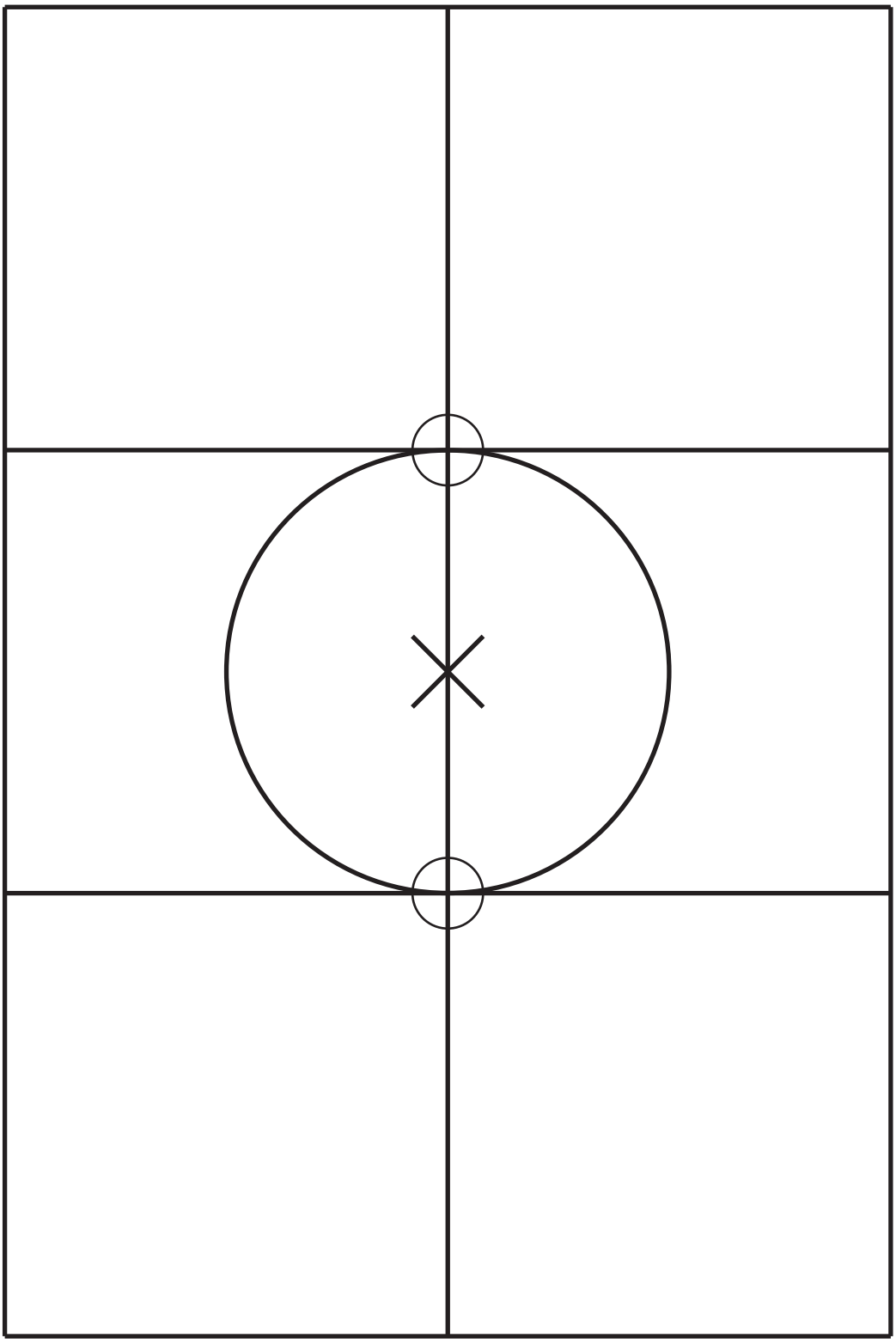
The following diagrams give examples for the sequence terms up to R^2=10. If the squared radius can be represented by an {obtuse (O, blue), right (R, red), acute (A, green)} triangle, one such representation is given. Illustrations of the degenerate cases excluded from the sequences are also given.

The legend at the bottom of the diagram provides the squared radius R^2, the radius R, the coordinates of the center of the cicumcircle, normalized such that 0<=x<=y, 0<=y<=1/2.

The last line of the legend shows
(number of grid points in the interior of the circumcircle) +
(number of grid points on the circle boundary) =
(total number of grid points covered).

Thanks to Hermann Jurksch for providing the list of circles and to Klaus Nagel for providing a program to create the illustrations.

Special Case, R^2=1/4 not representable
by circumcircle of 3 points of square lattice



$R^2 = 1/4 = 0.25000$

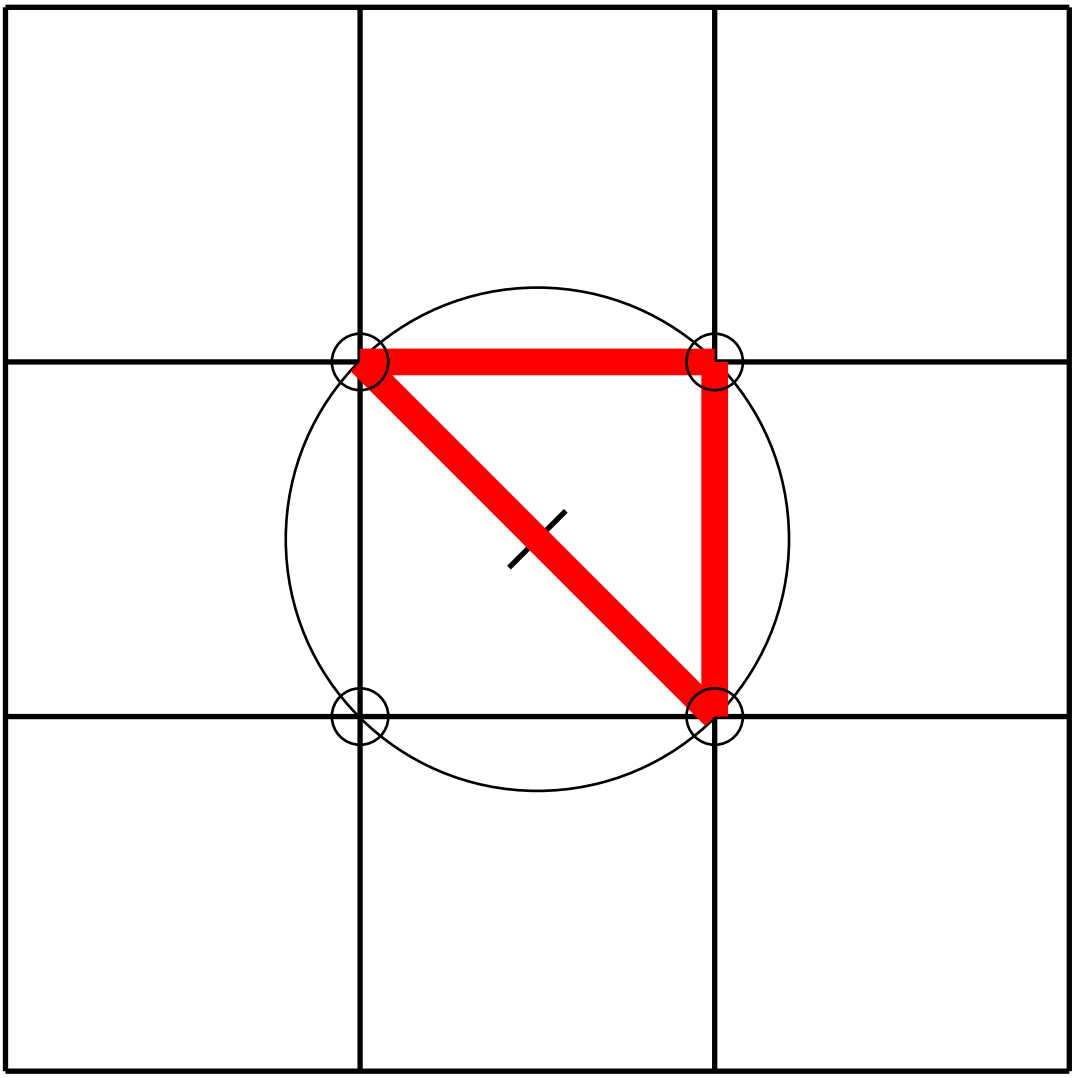
$R = 0.500000$

$X = 0/1$

$Y = 1/2$

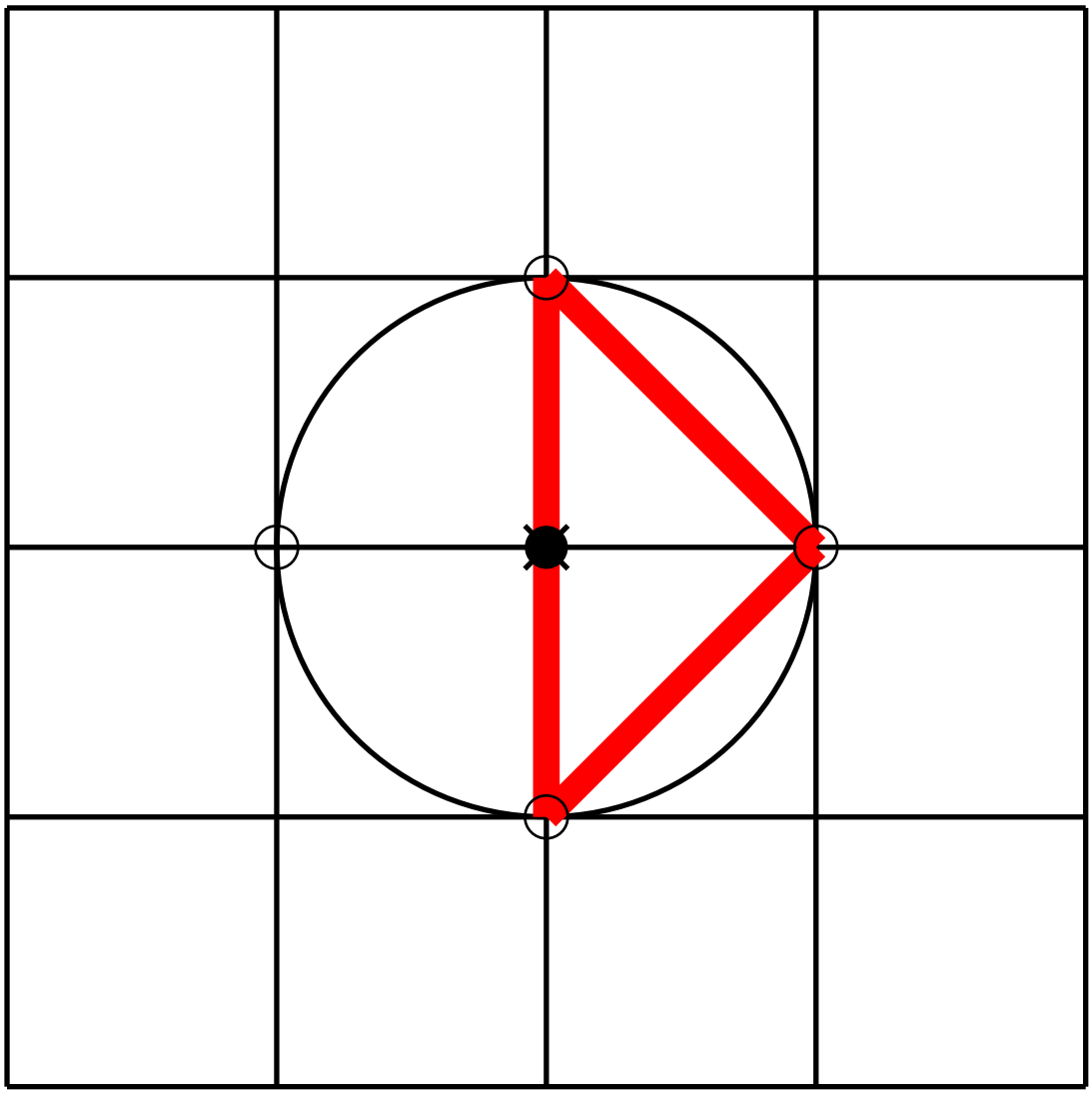
$0 + 2 = 2$

A192493(1) = 1, A192494(1) = 2
Triangles: R



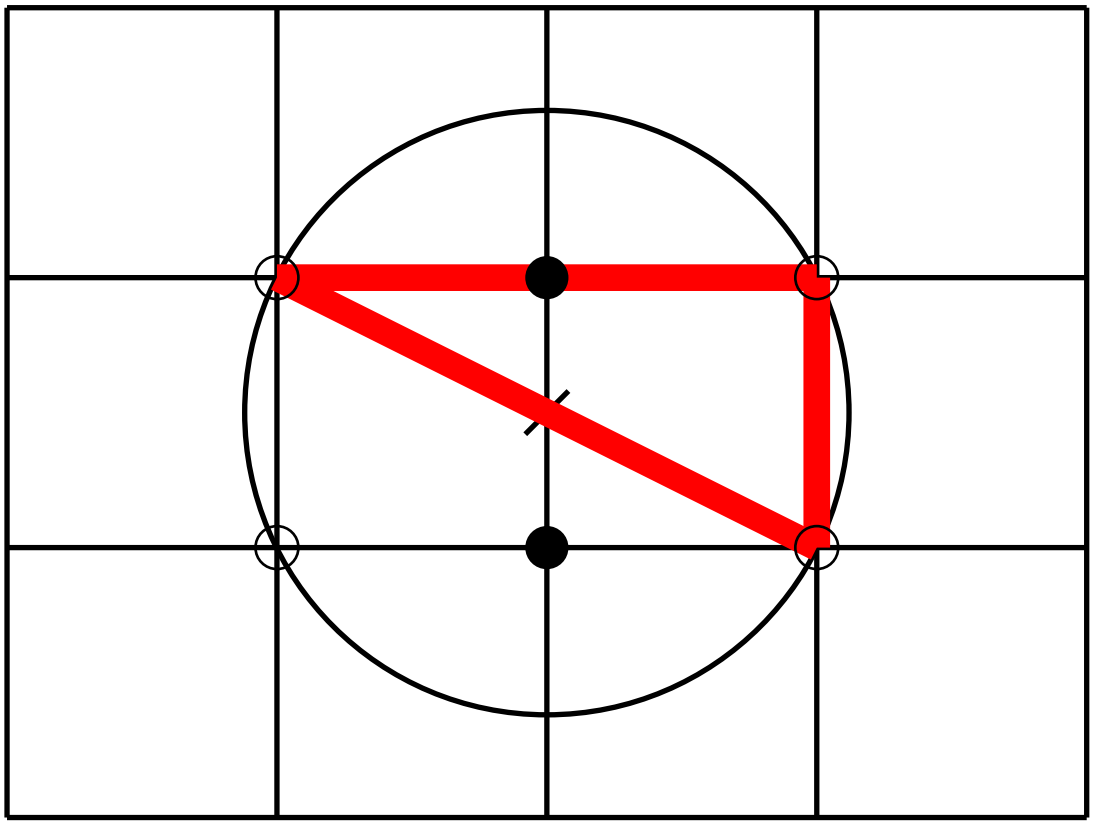
$R^2 = 1/2 = 0.50000$
$R = 0.707107$
$X = 1/2$
$Y = 1/2$
$0 + 4 = 4$

A192493(2) = 1, A192494(2) = 2
Triangles: R



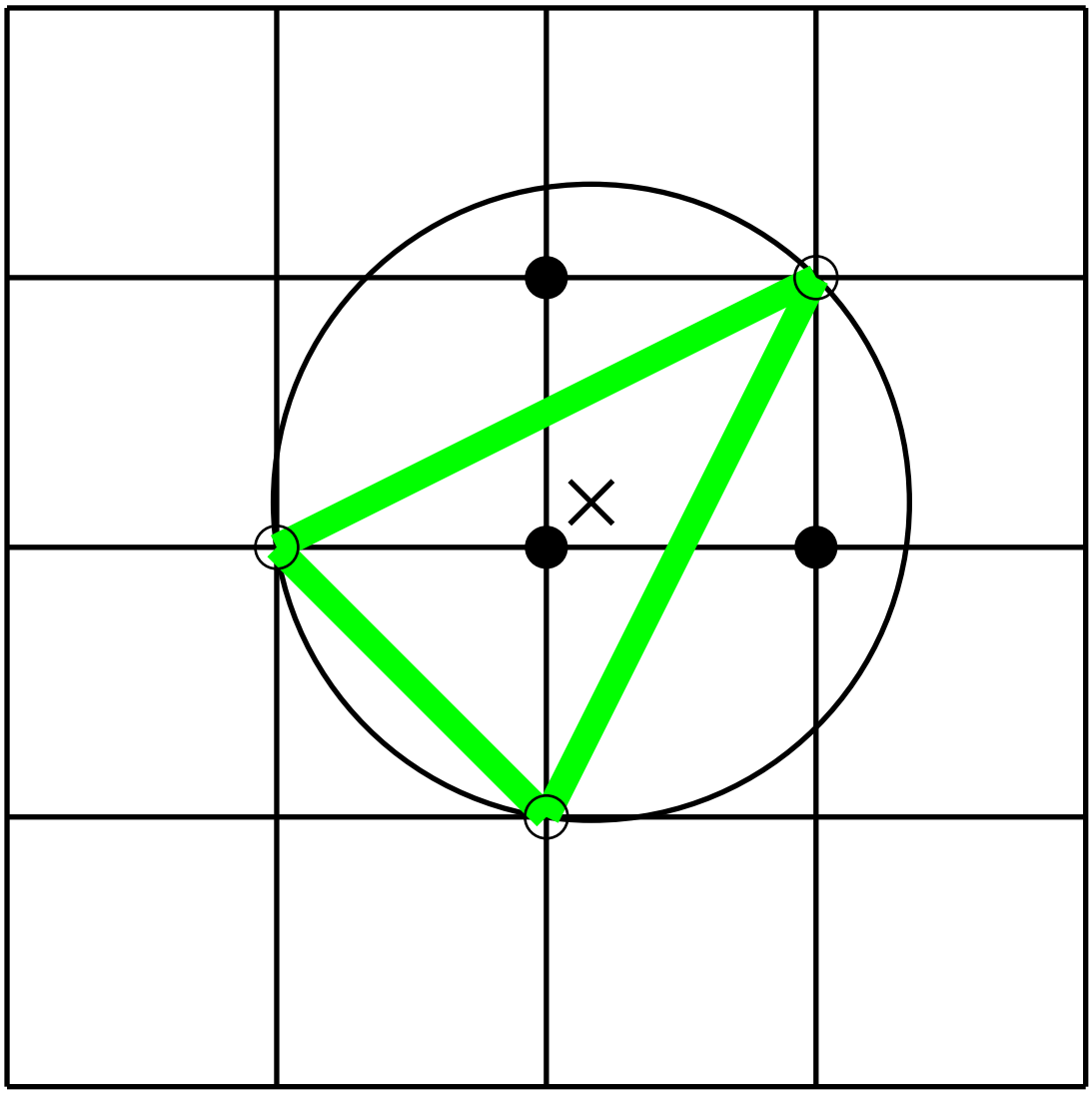
R^2 = 1 / 1 = 1.00000
R = 1.000000
X = 0 / 1
Y = 0 / 1
1 + 4 = 5

A192493(3) = 5, A192494(3) = 4
Triangles: R



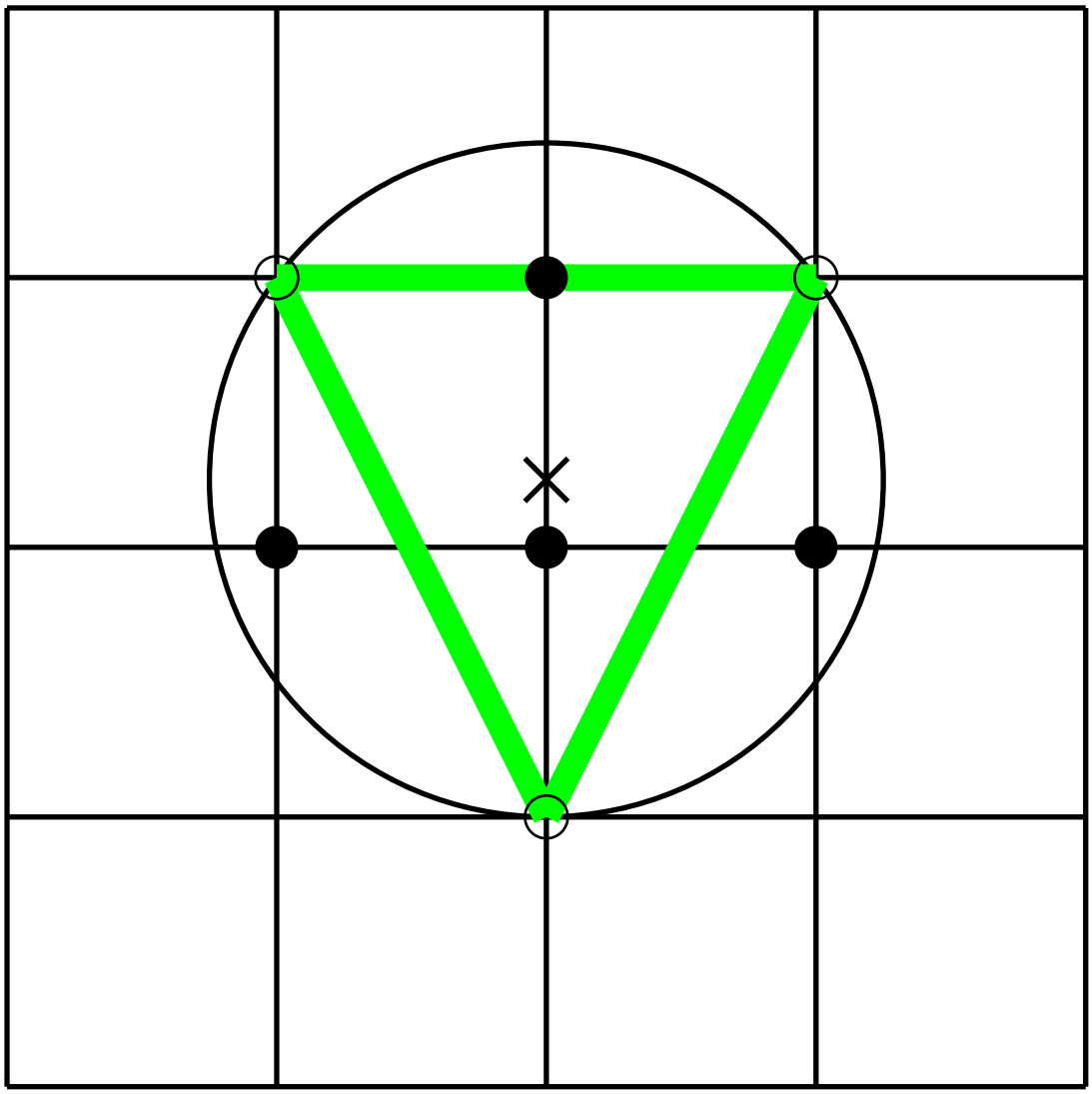
R^2 = 5/4 = 1.25000
R = 1.118034
X = 0/1
Y = 1/2
2 + 4 = 6

A192493(4) = 25, A192494(4) = 18
Triangles: A



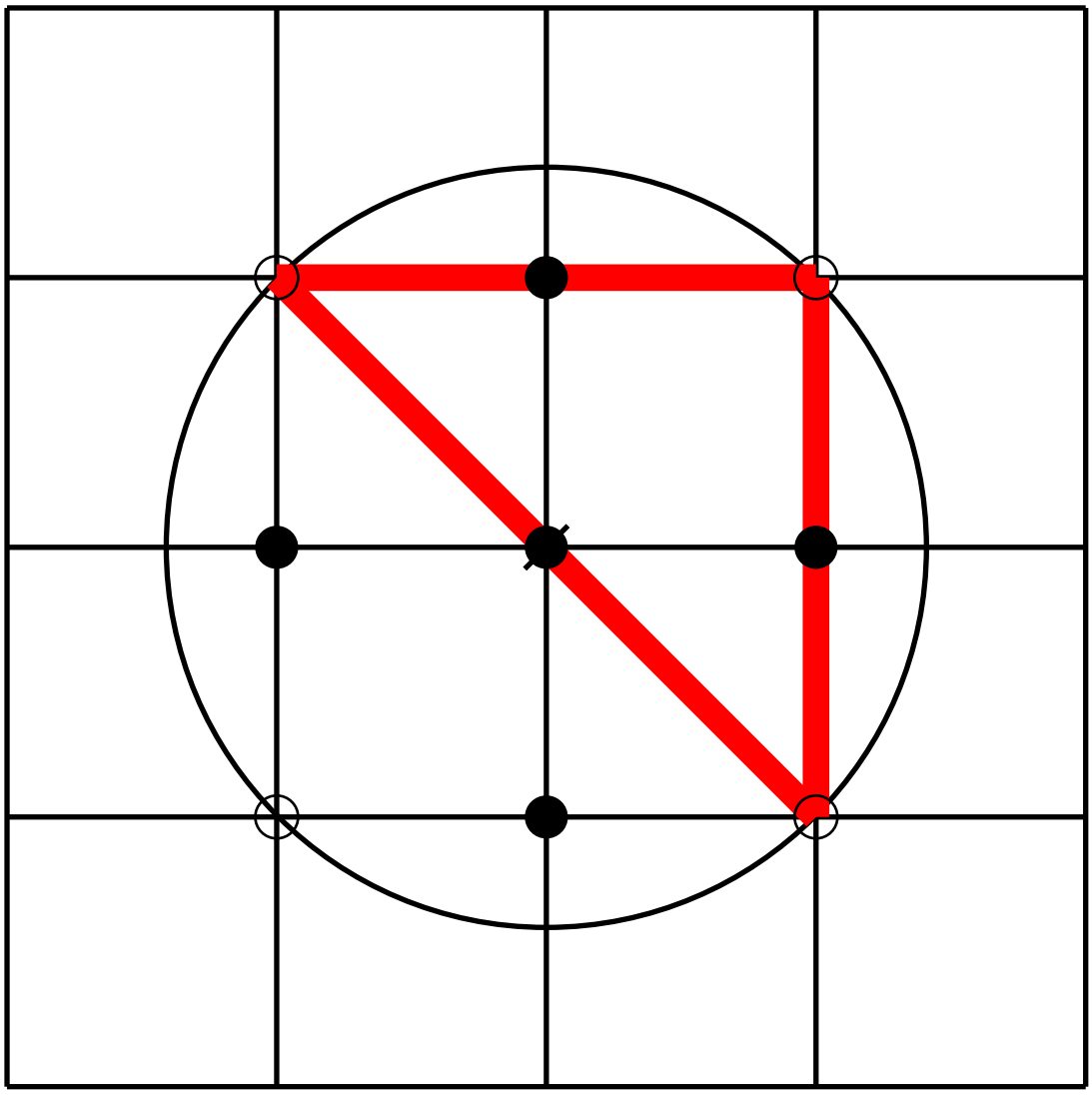
R^2 = 25/18 = 1.38889
R = 1.178511
X = 1/6
Y = 1/6
3 + 3 = 6

A192493(5) = 25, A192494(5) = 16
Triangles: A



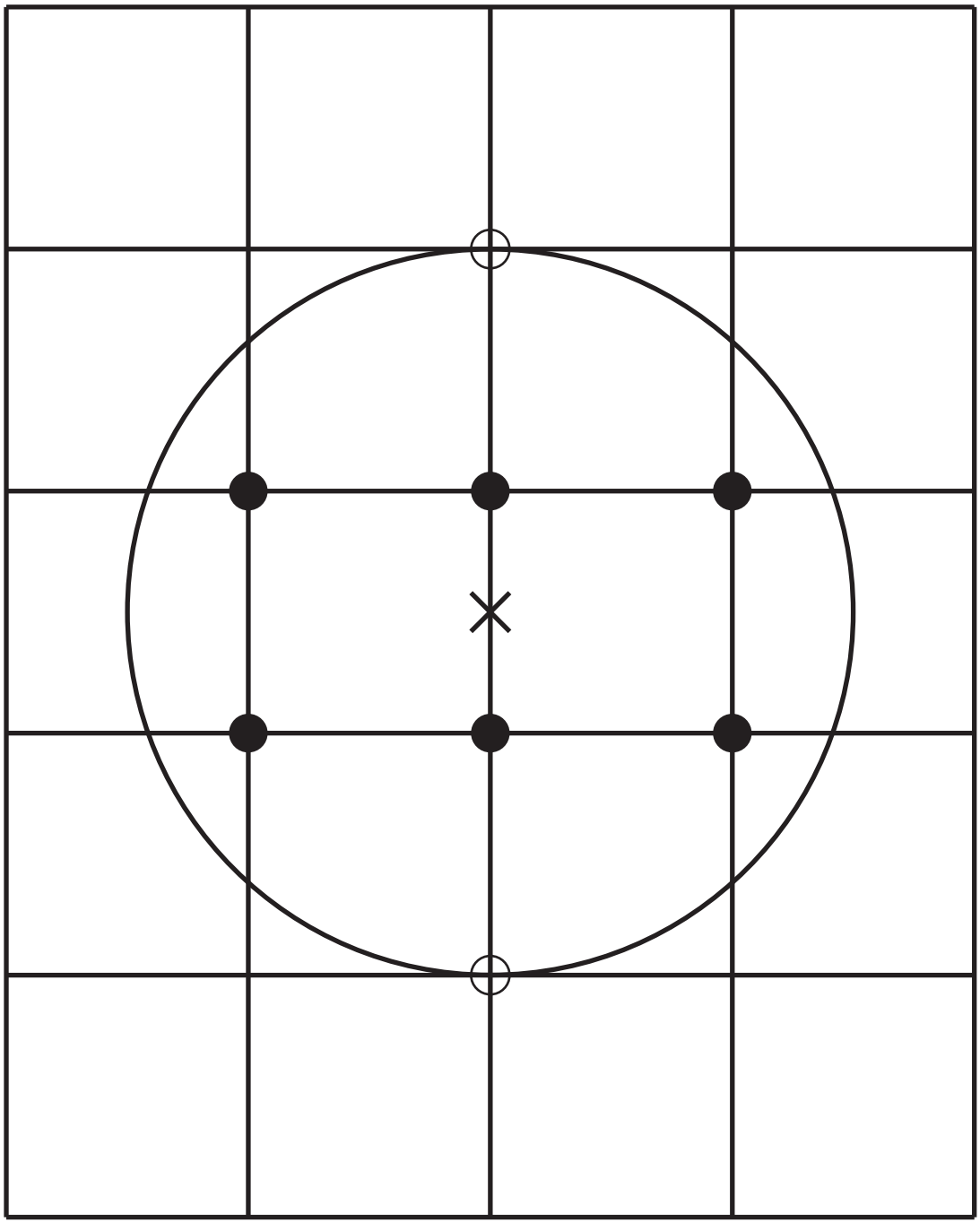
R^2 = 25/16 = 1.56250
R = 1.250000
X = 0/1
Y = 1/4
4 + 3 = 7

A192493(6) = 2, A192494(6) = 1
Triangles: R



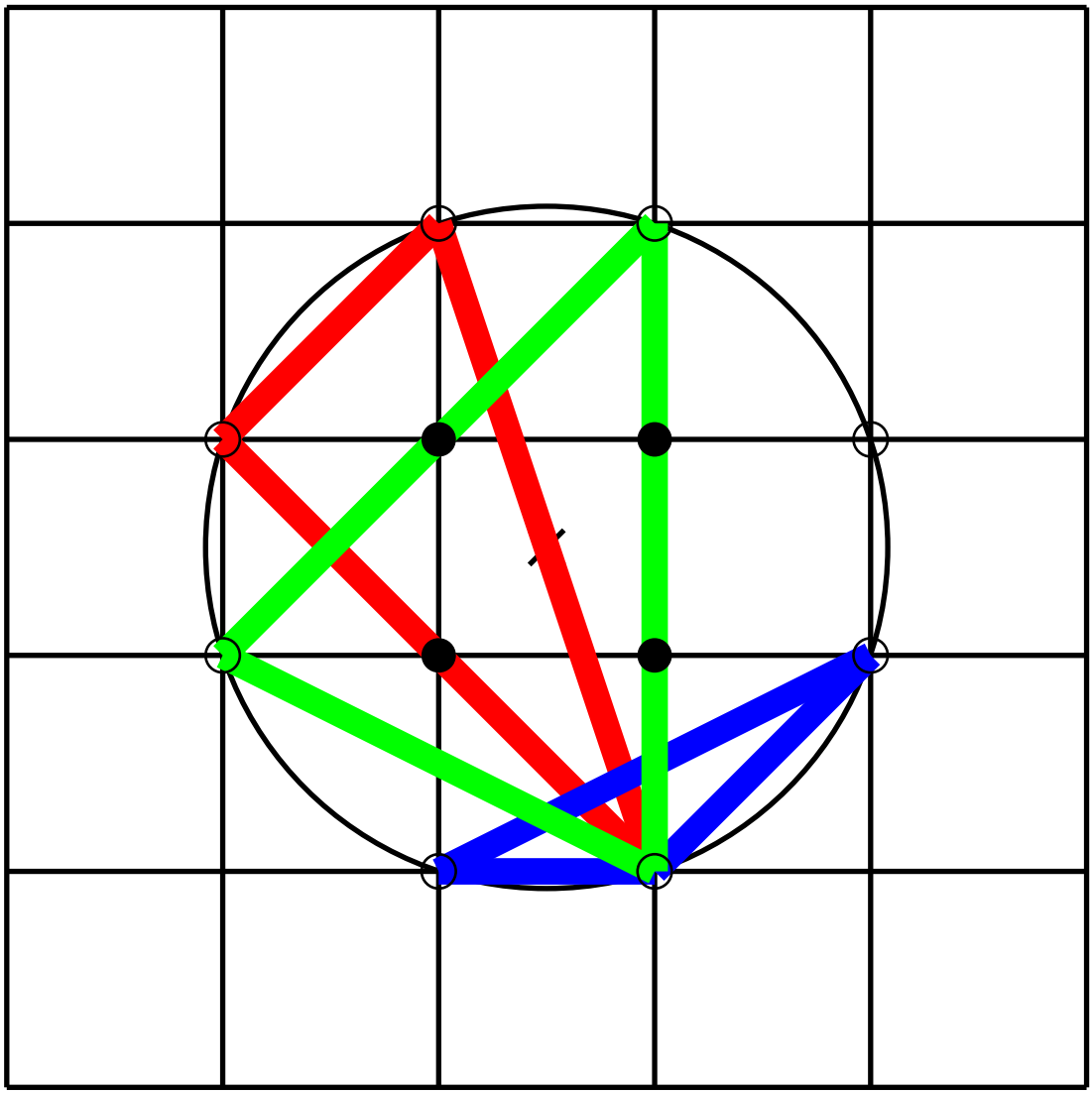
R^2 = 2/1 = 2.00000
R = 1.414214
X = 0/1
Y = 0/1
5 + 4 = 9

Special Case, R^2=9/4 not representable
by circumcircle of 3 points of square lattice



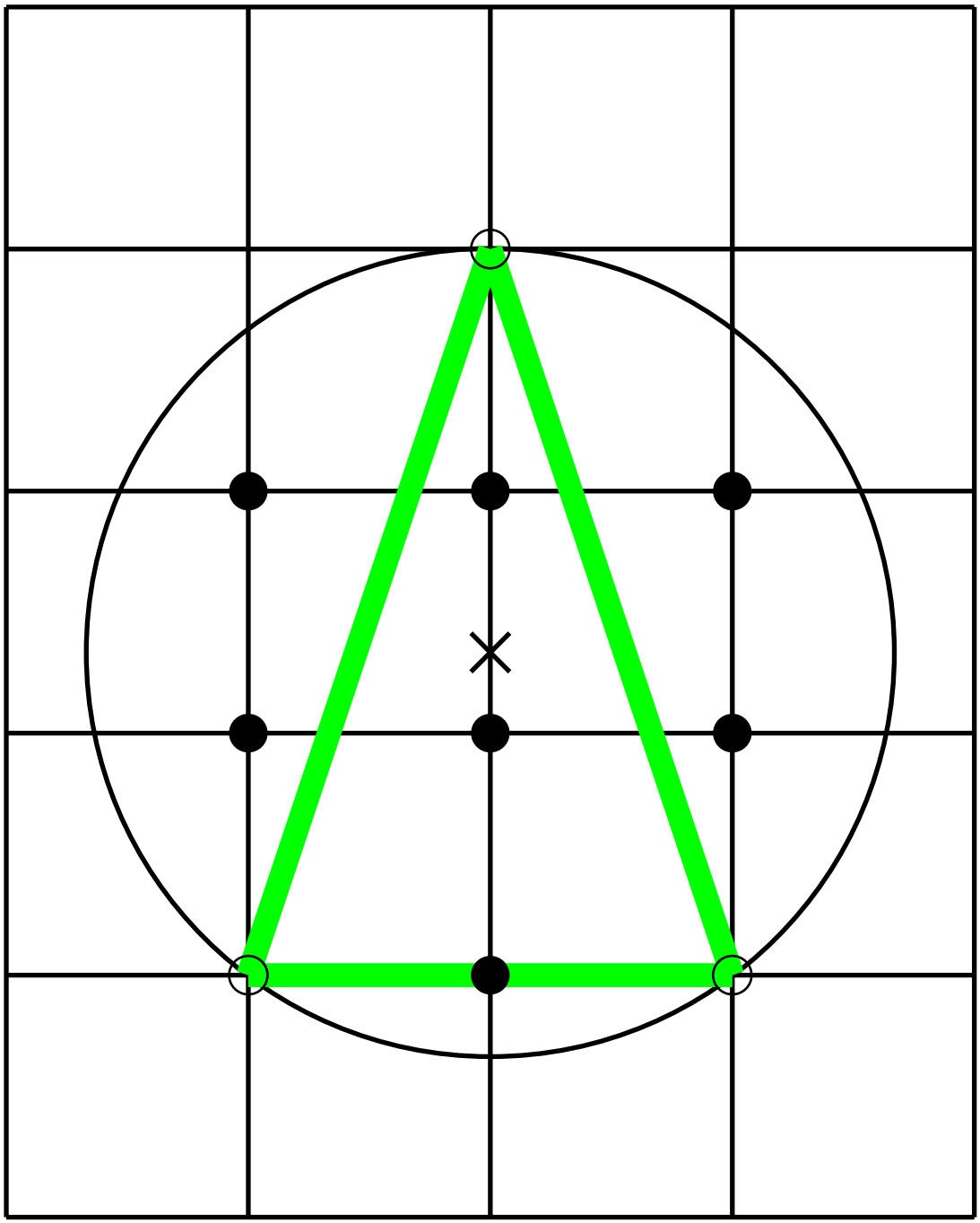
R^2 = 9/4 = 2.25000
R = 1.500000
X = 0/1
Y = 1/2
6 + 2 = 8

A192493(7) = 5, A192494(7) = 2
Triangles: O R A



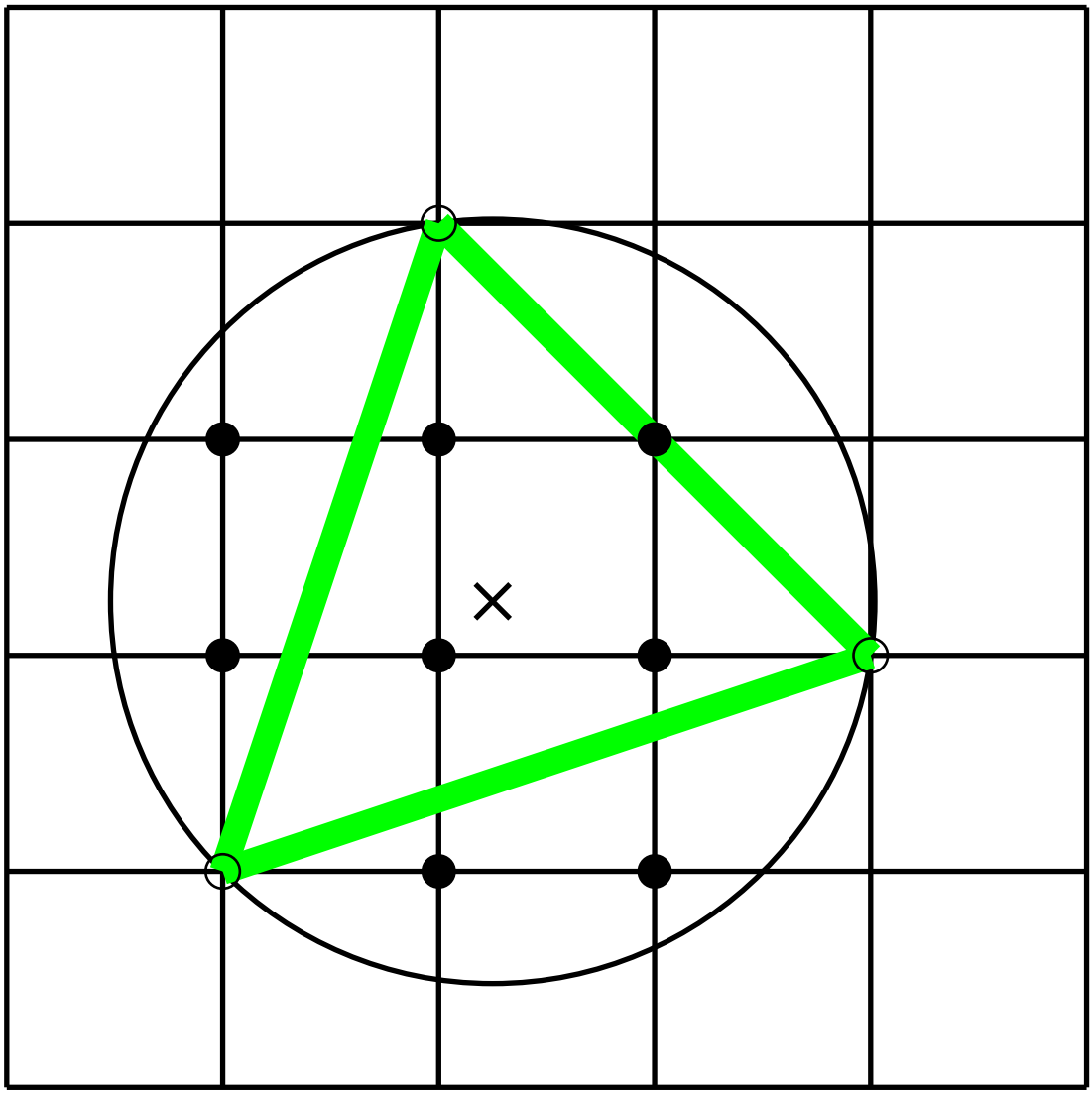
$R^2 = 5/2 = 2.50000$
$R = 1.581139$
$X = 1/2$
$Y = 1/2$
$4 + 8 = 12$

A192493(8) = 25, A192494(8) = 9
Triangles: A



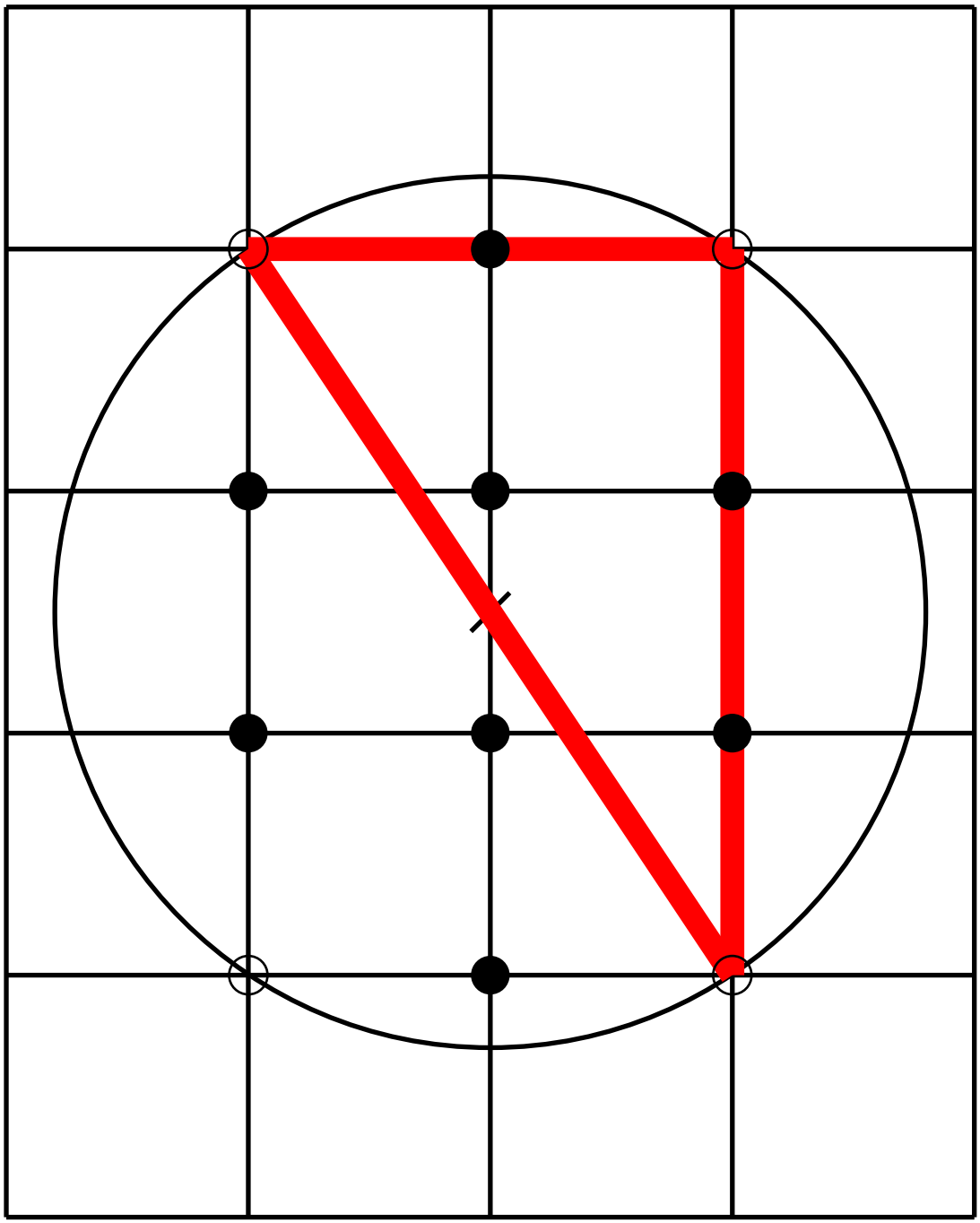
R^2 = 25/9 = 2.77778
R = 1.666667
X = 0/1
Y = 1/3
7 + 3 = 10

A192493(9) = 25, A192494(9) = 8
Triangles: A



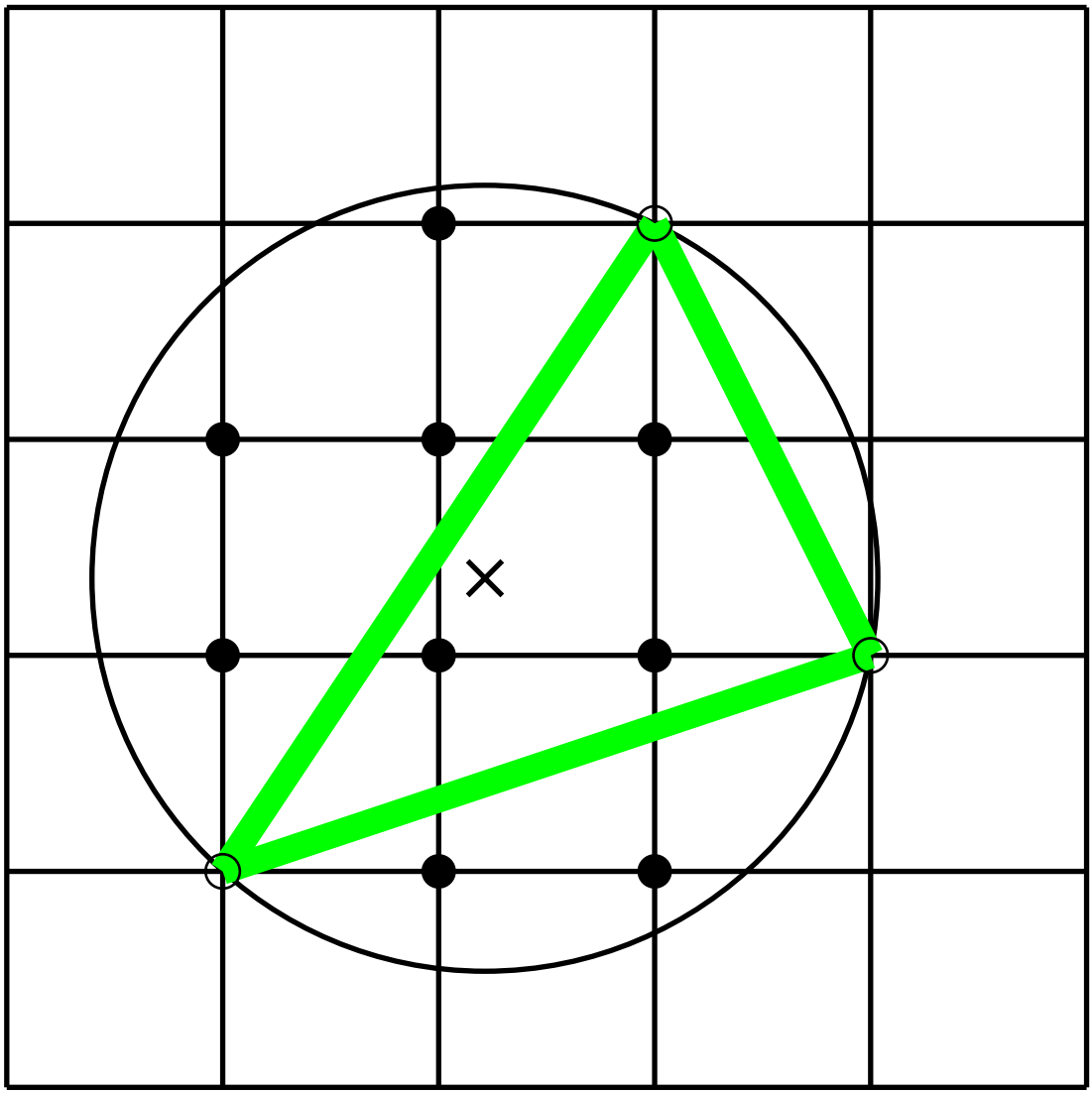
R^2 = 25/8 = 3.12500
R = 1.767767
X = 1/4
Y = 1/4
8 + 3 = 11

A192493(10) = 13, A192494(10) = 4
Triangles: R



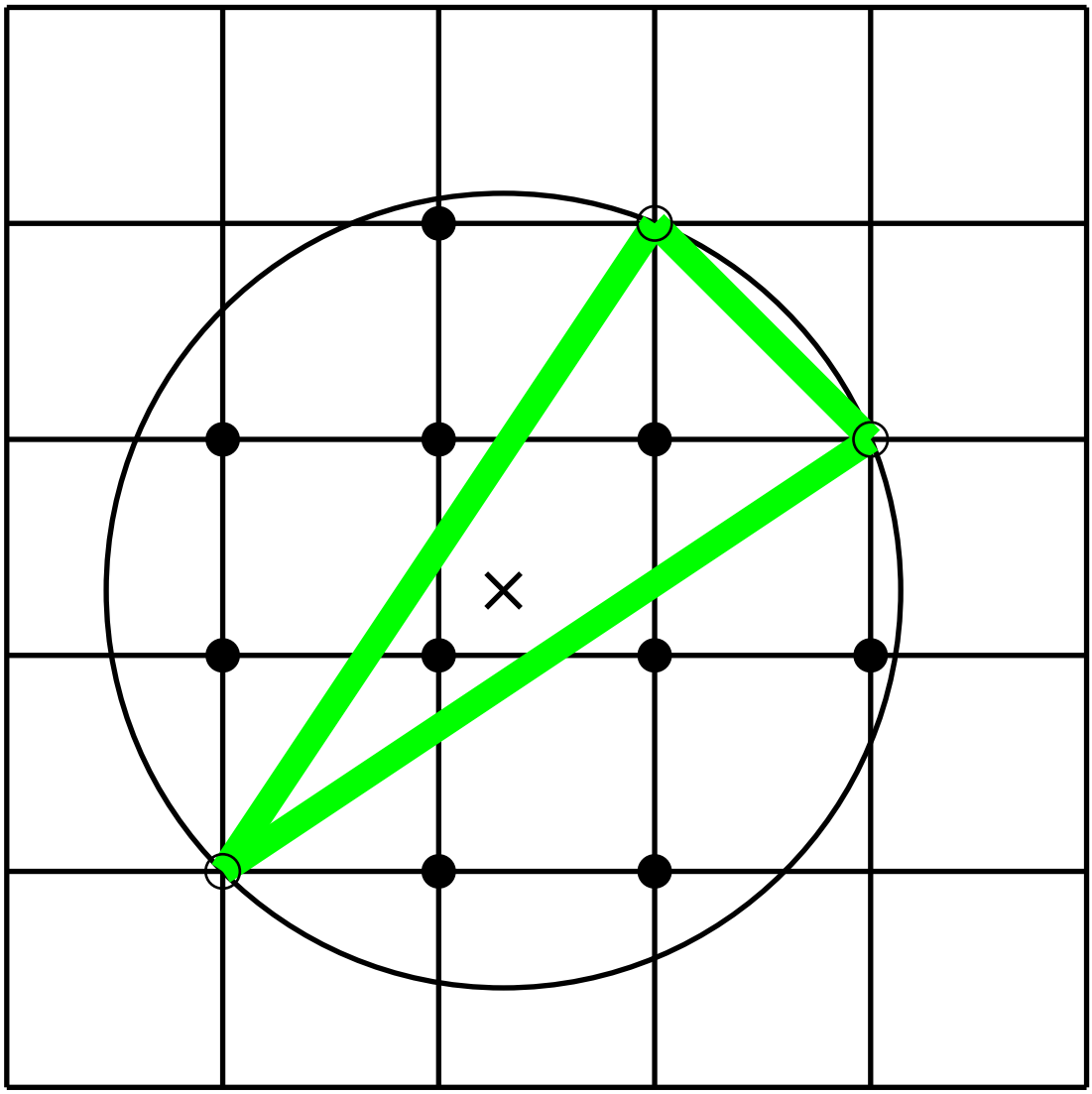
R^2 = 13/4 = 3.25000
R = 1.802776
X = 0/1
Y = 1/2
8 + 4 = 12

A192493(11) = 325, A192494(11) = 98
Triangles: A



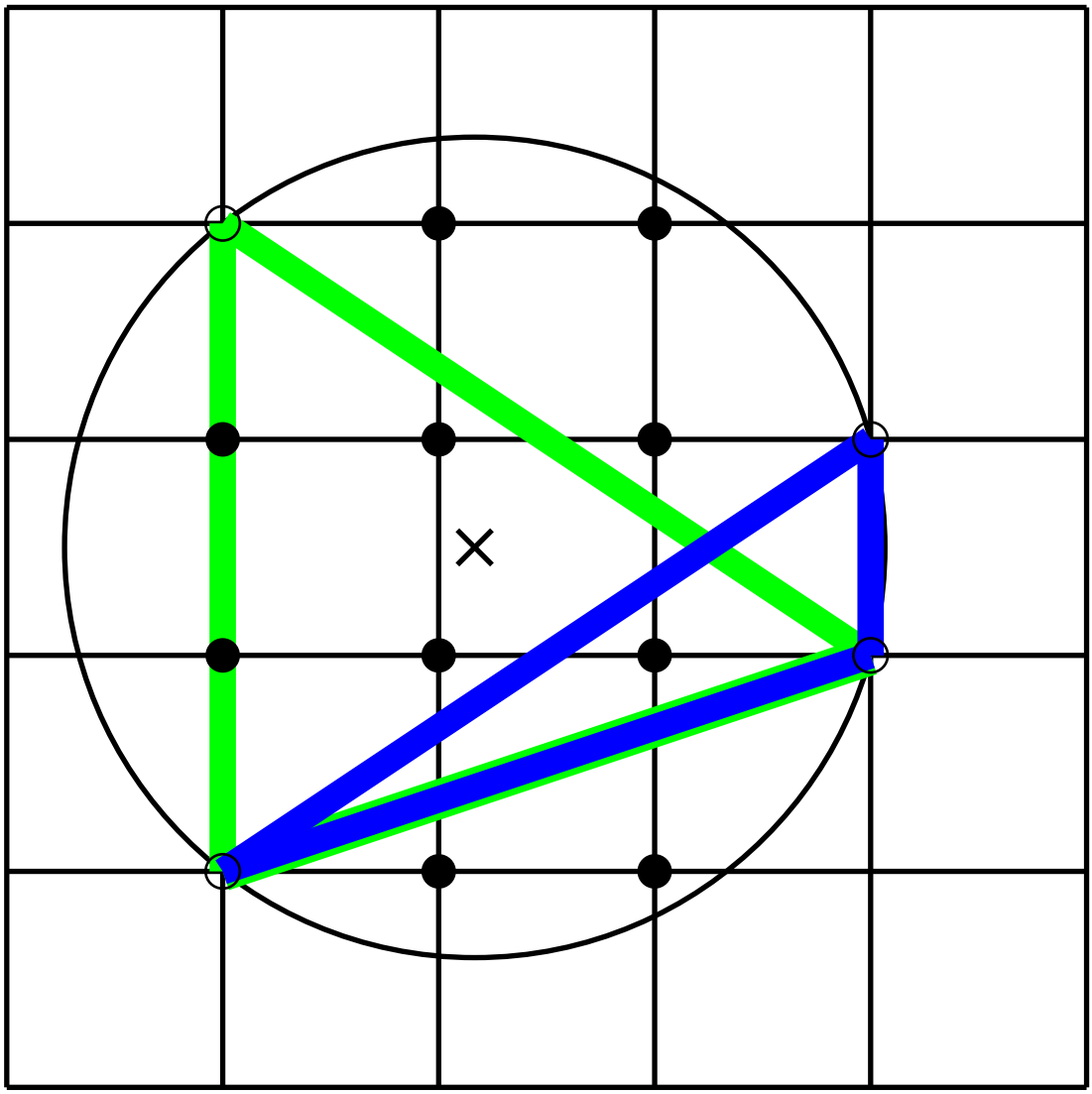
R^2 = 325/ 98 = 3.31633
R = 1.821078
X = 3/ 14
Y = 5/ 14
9 + 3 = 12

A192493(12) = 169, A192494(12) = 50
Triangles: A



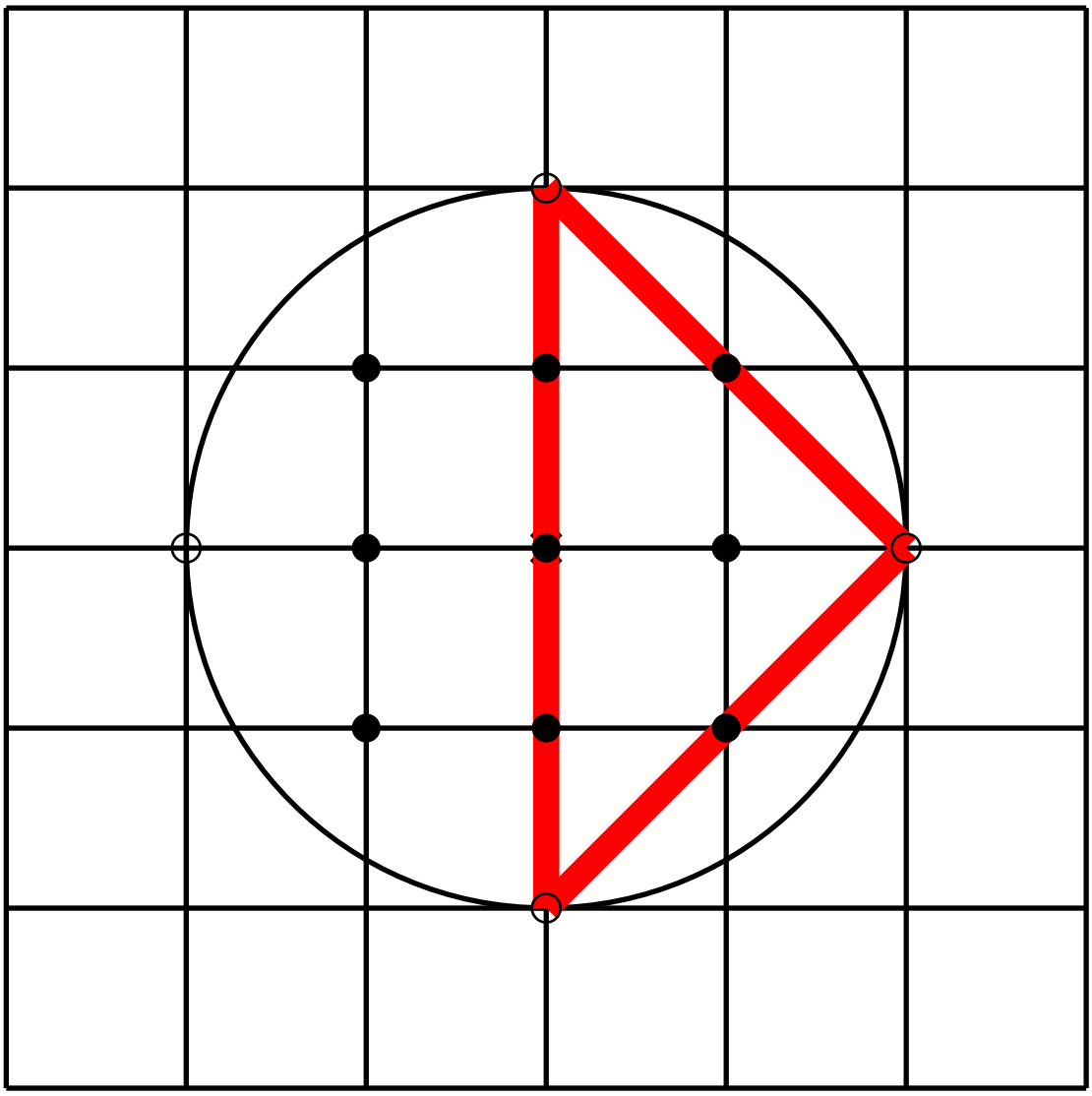
$R^2 = 169/50 = 3.38000$
$R = 1.838478$
$X = 3/10$
$Y = 3/10$
$10 + 3 = 13$

A192493(13) = 65, A192494(13) = 18
Triangles: O A



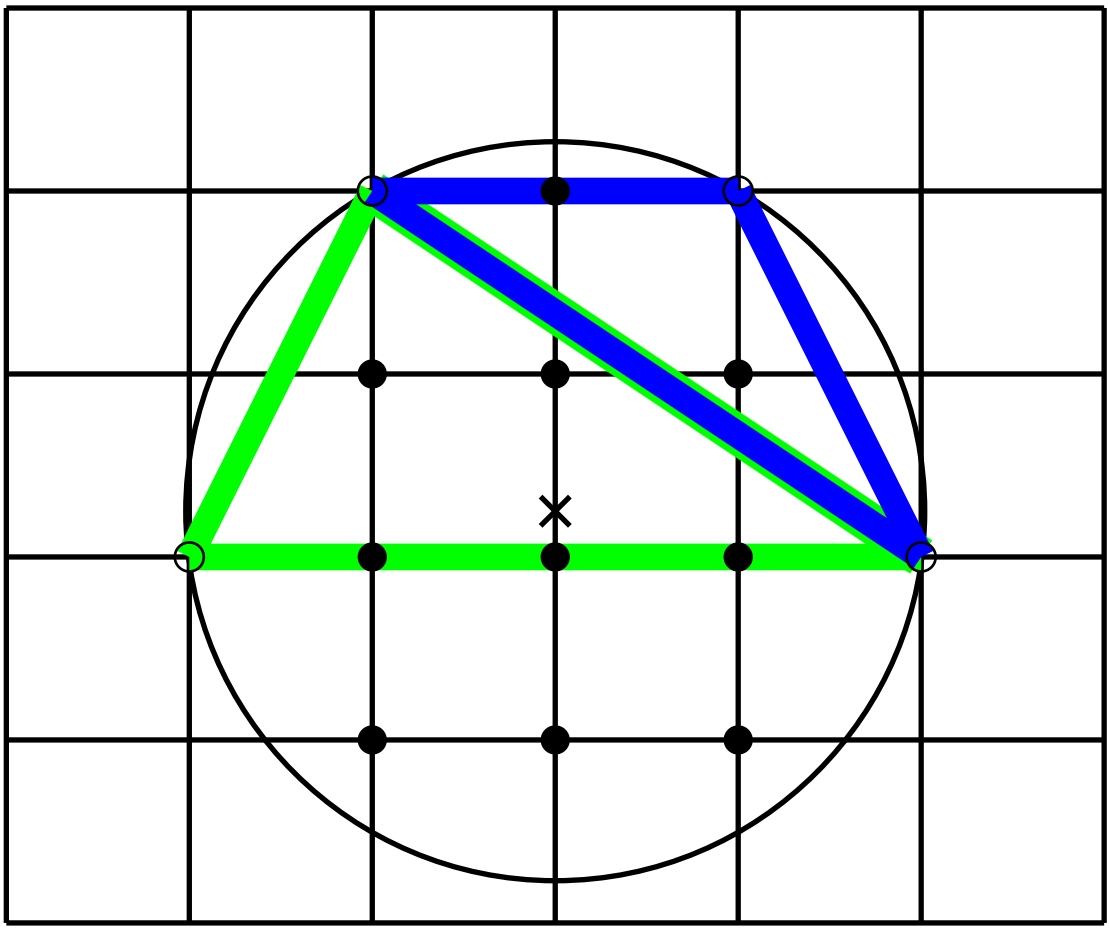
R^2 = 65/ 18 = 3.61111
R = 1.900292
X = 1/6
Y = 1/2
10 + 4 = 14

A192493(14) = 4, A192494(14) = 1
Triangles: R



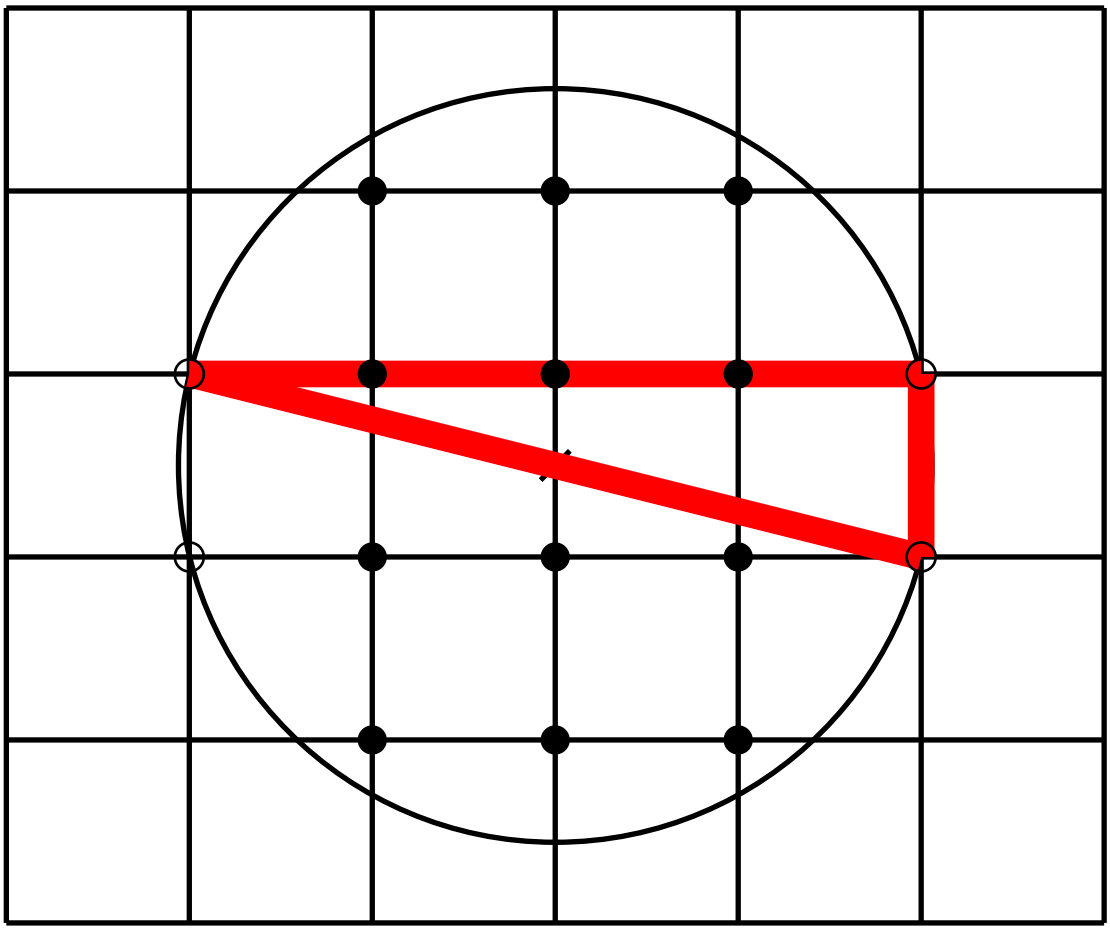
R^2 = 4 / 1 = 4.00000
R = 2.000000
X = 0 / 1
Y = 0 / 1
9 + 4 = 13

A192493(15) = 65, A192494(15) = 16
Triangles: O A



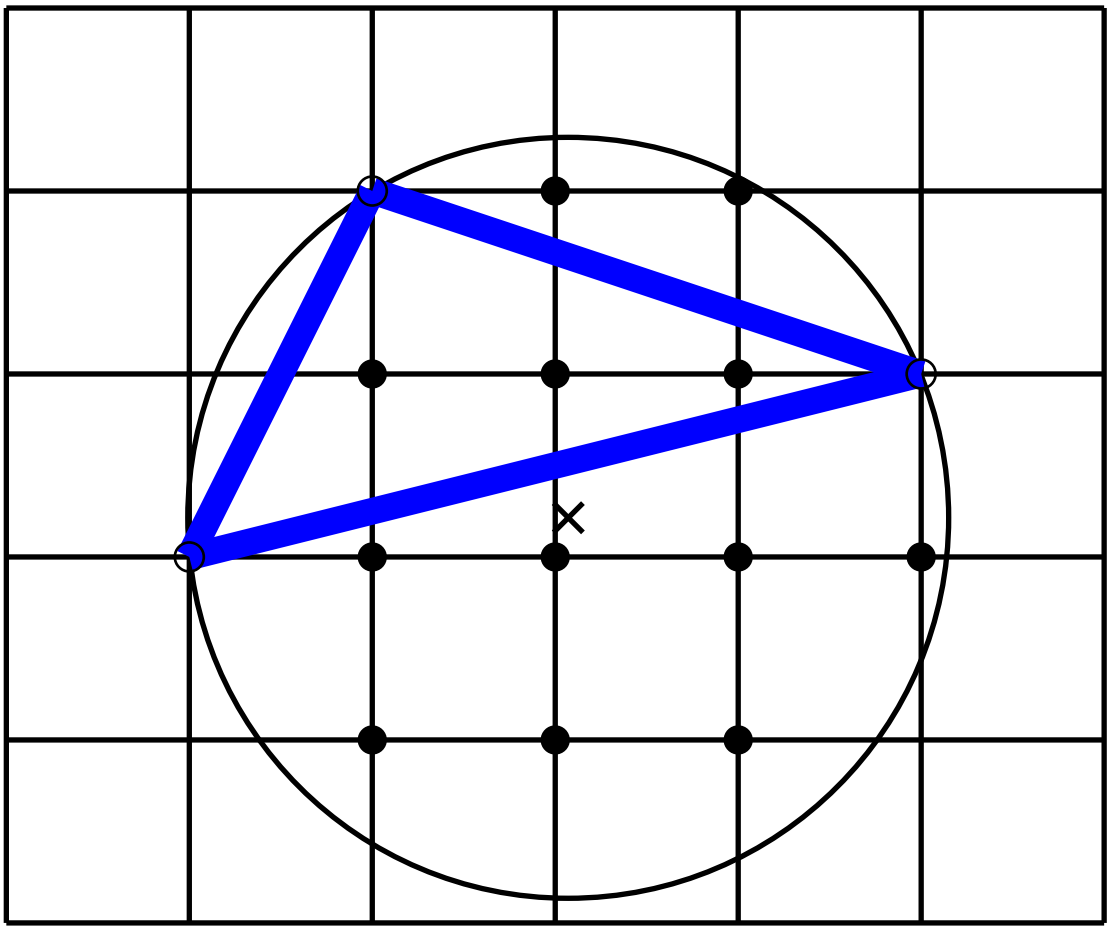
R^2 = 65/16 = 4.06250
R = 2.015564
X = 0/1
Y = 1/4
10 + 4 = 14

A192493(16) = 17, A192494(16) = 4
Triangles: R



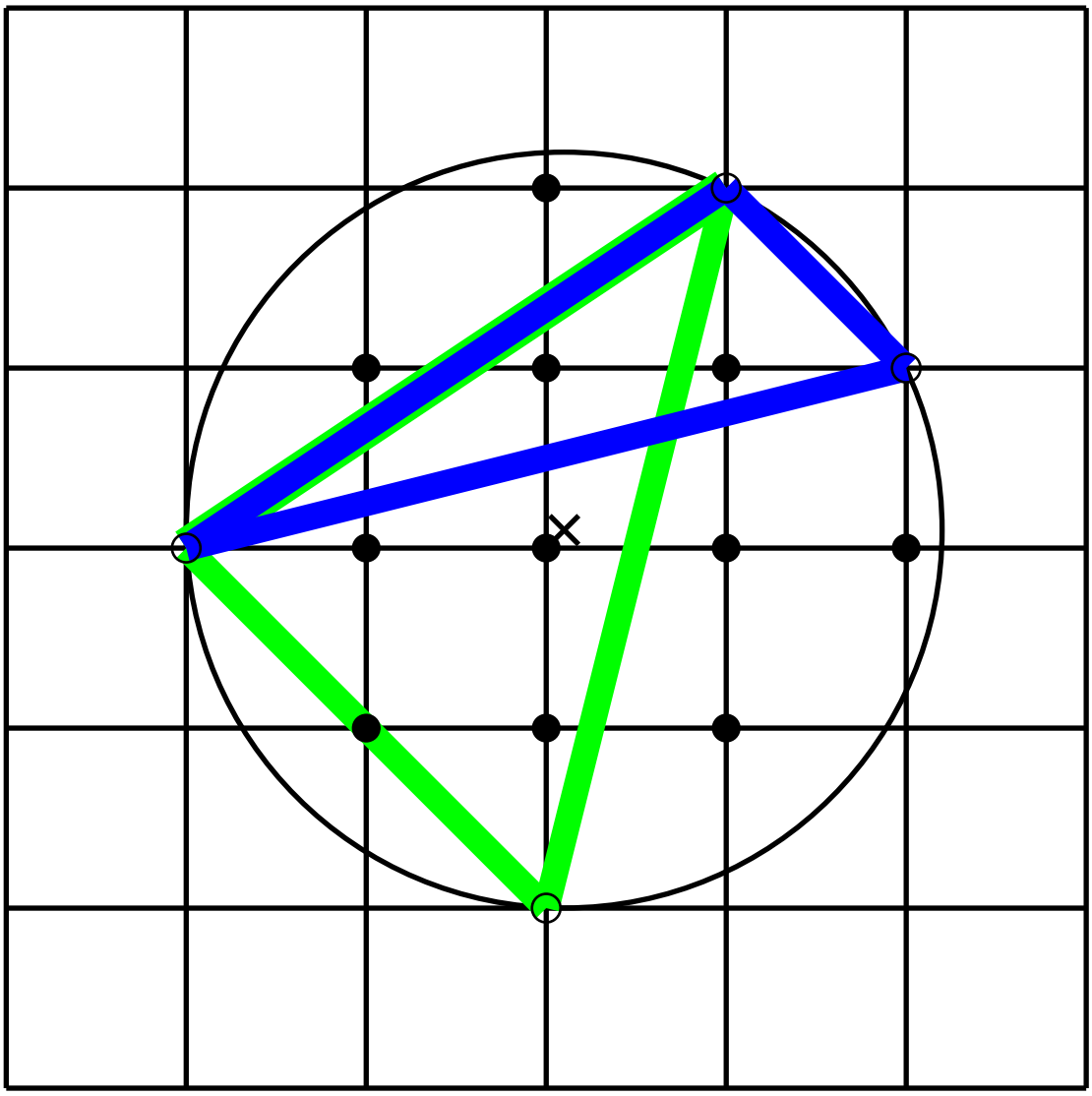
R^2 = 17 / 4 = 4.25000
R = 2.061553
X = 0 / 1
Y = 1 / 2
12 + 4 = 16

A192493(17) = 425, A192494(17) = 98
Triangles: O



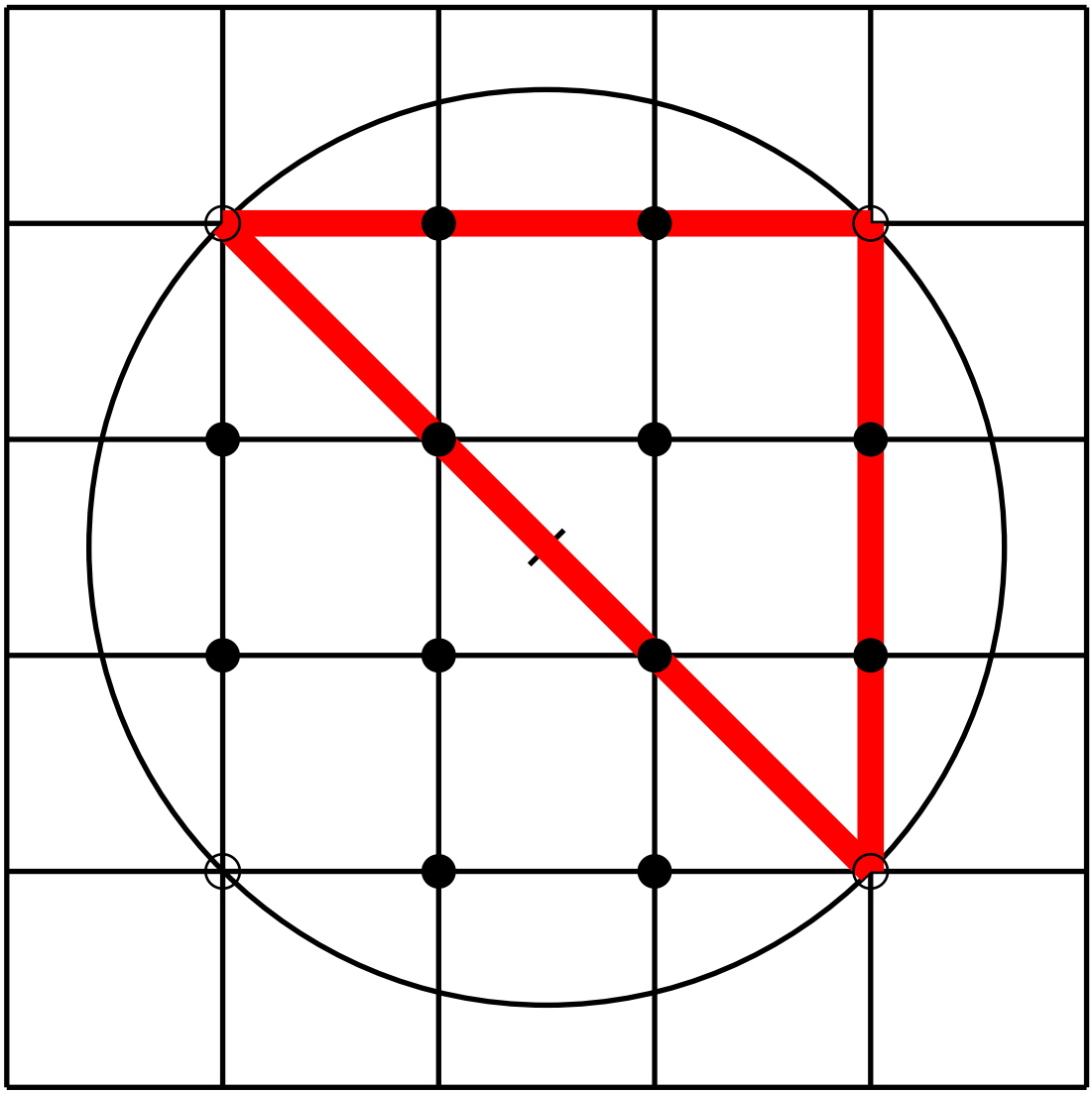
$R^2 = 425/98 = 4.33673$
$R = 2.082483$
$X = 1/14$
$Y = 3/14$
$12 + 3 = 15$

A192493(18) = 221, A192494(18) = 50
Triangles: O A



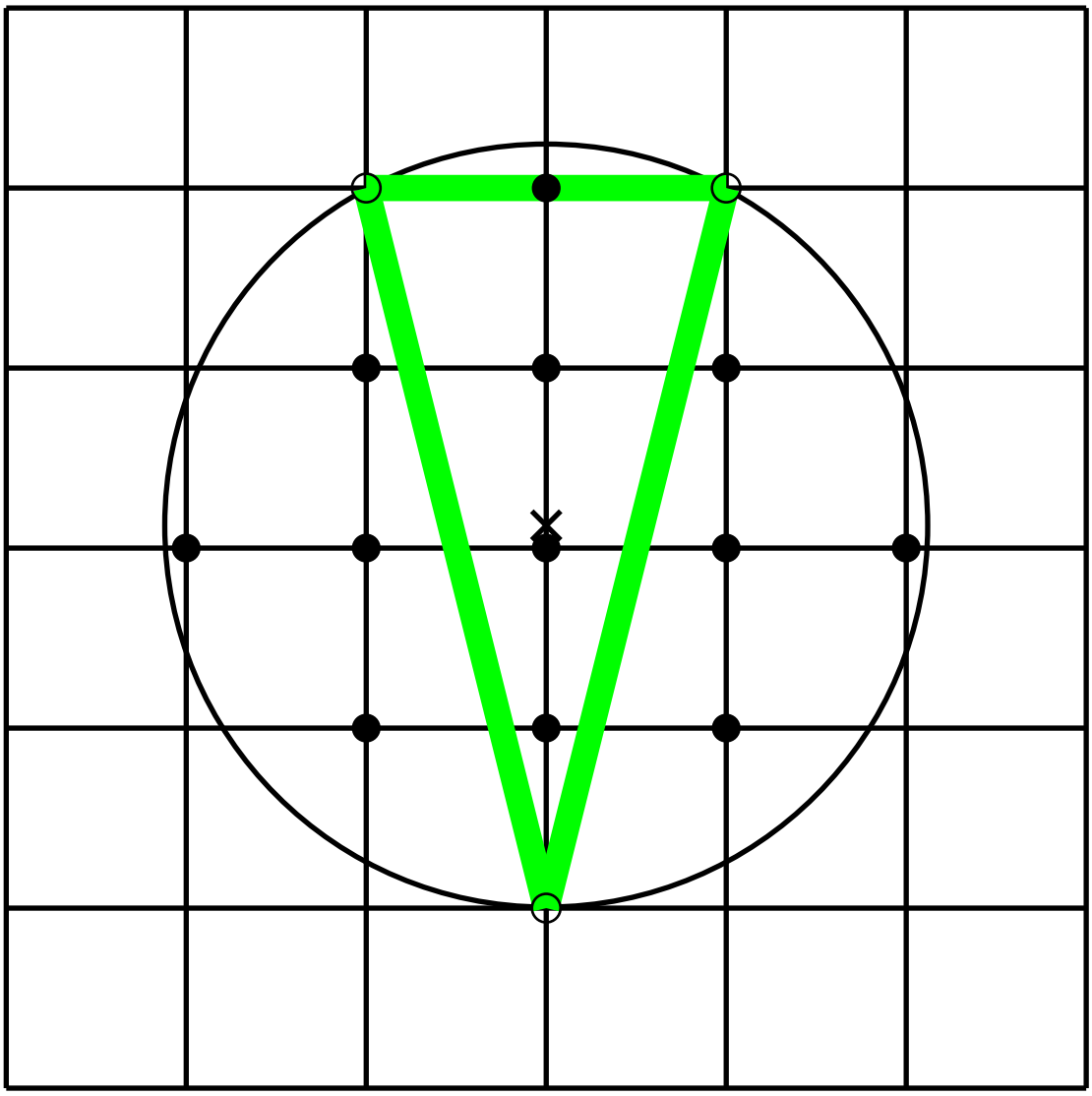
R^2 = 221 / 50 = 4.42000
R = 2.102380
X = 1 / 10
Y = 1 / 10
11 + 4 = 15

A192493(19) = 9, A192494(19) = 2
Triangles: R



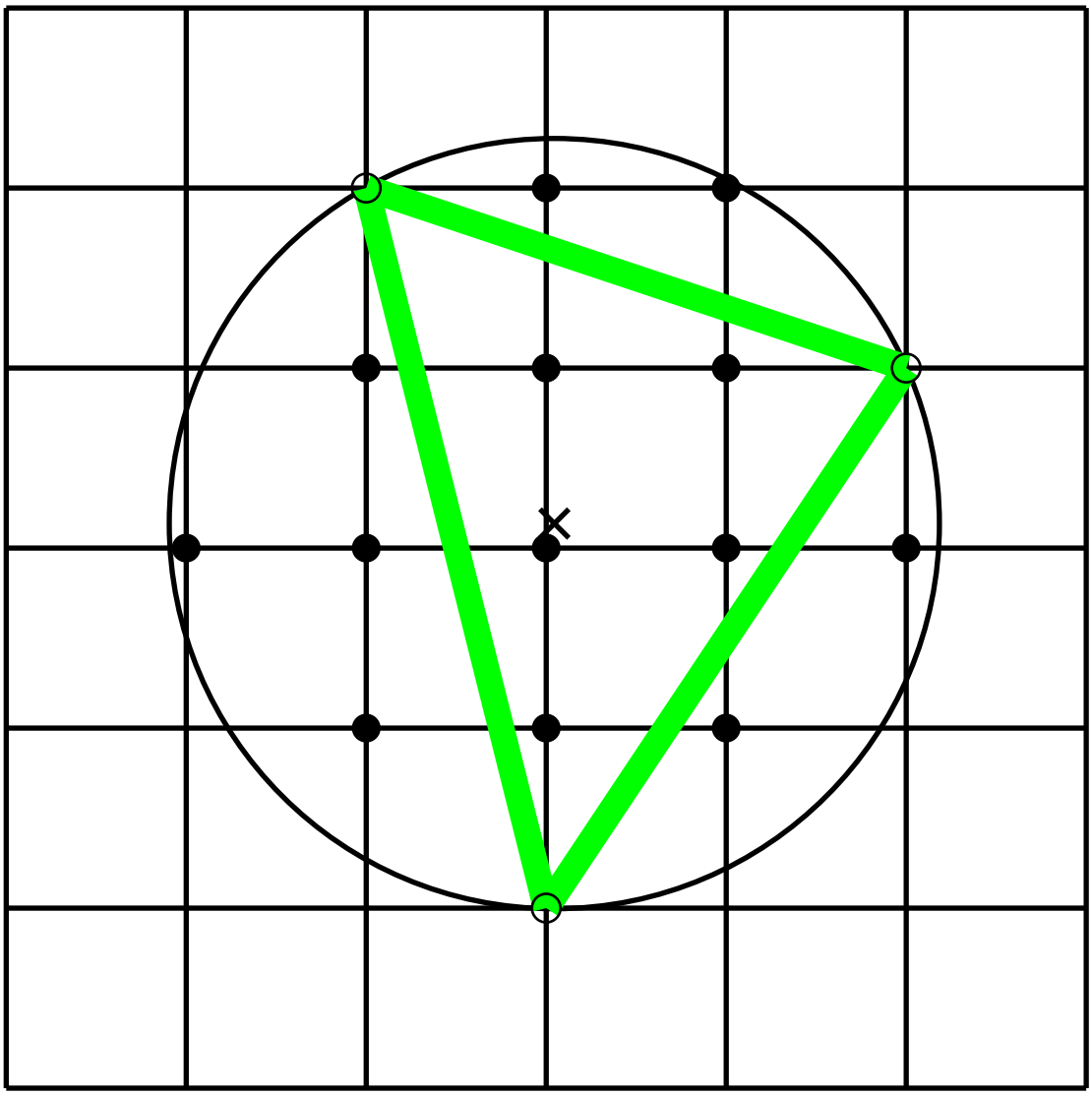
$R^2 = 9/2 = 4.500000$
$R = 2.121320$
$X = 1/2$
$Y = 1/2$
$12 + 4 = 16$

A192493(20) = 289, A192494(20) = 64
Triangles: A



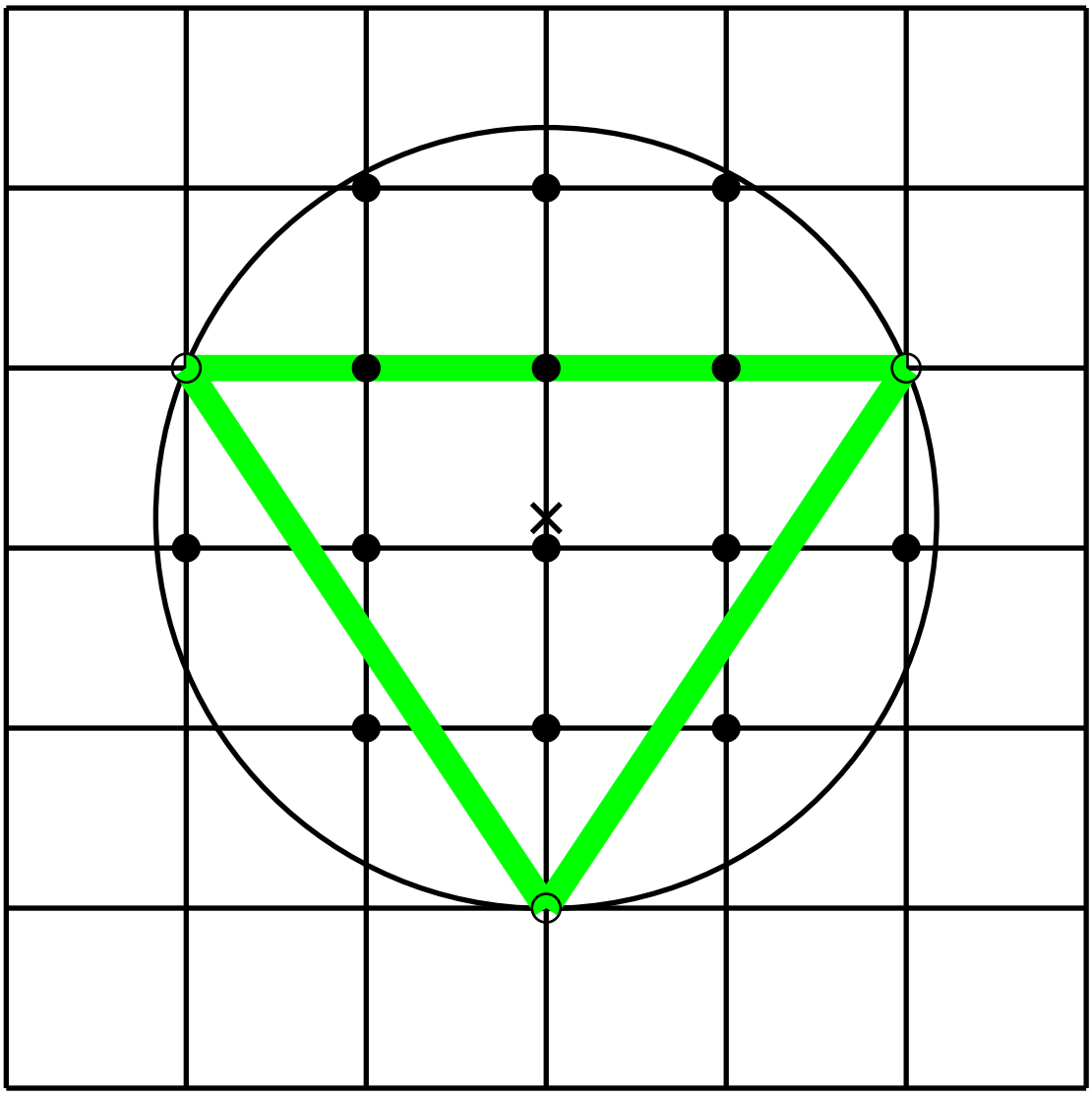
R^2 = 289/64 = 4.51562
R = 2.125000
X = 0/1
Y = 1/8
12 + 3 = 15

A192493(21) = 1105, A192494(21) = 242
Triangles: A



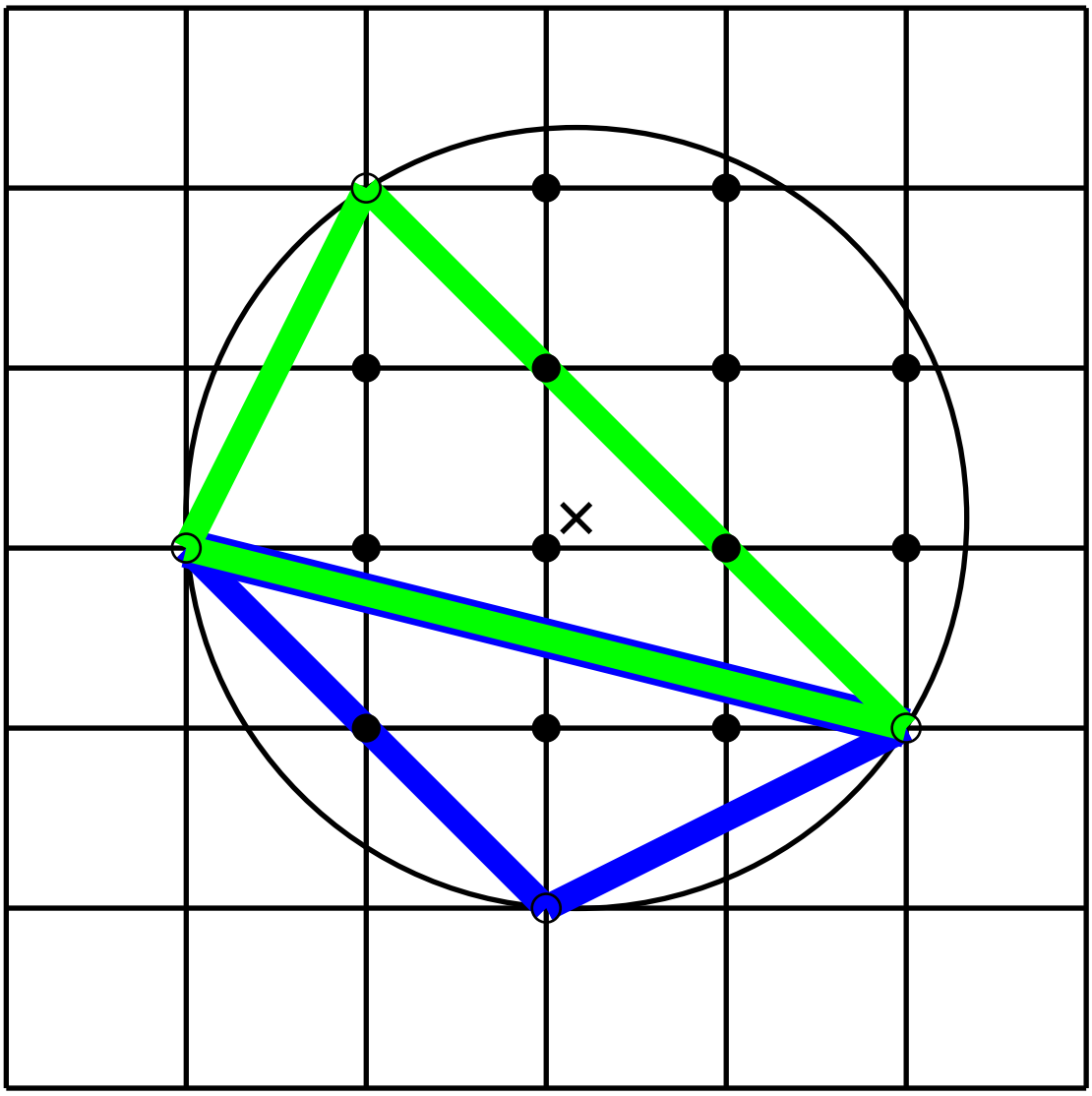
R^2 = 1105/242 = 4.56612
R = 2.136847
X = 1/22
Y = 3/22
13 + 3 = 16

A192493(22) = 169, A192494(22) = 36
Triangles: A



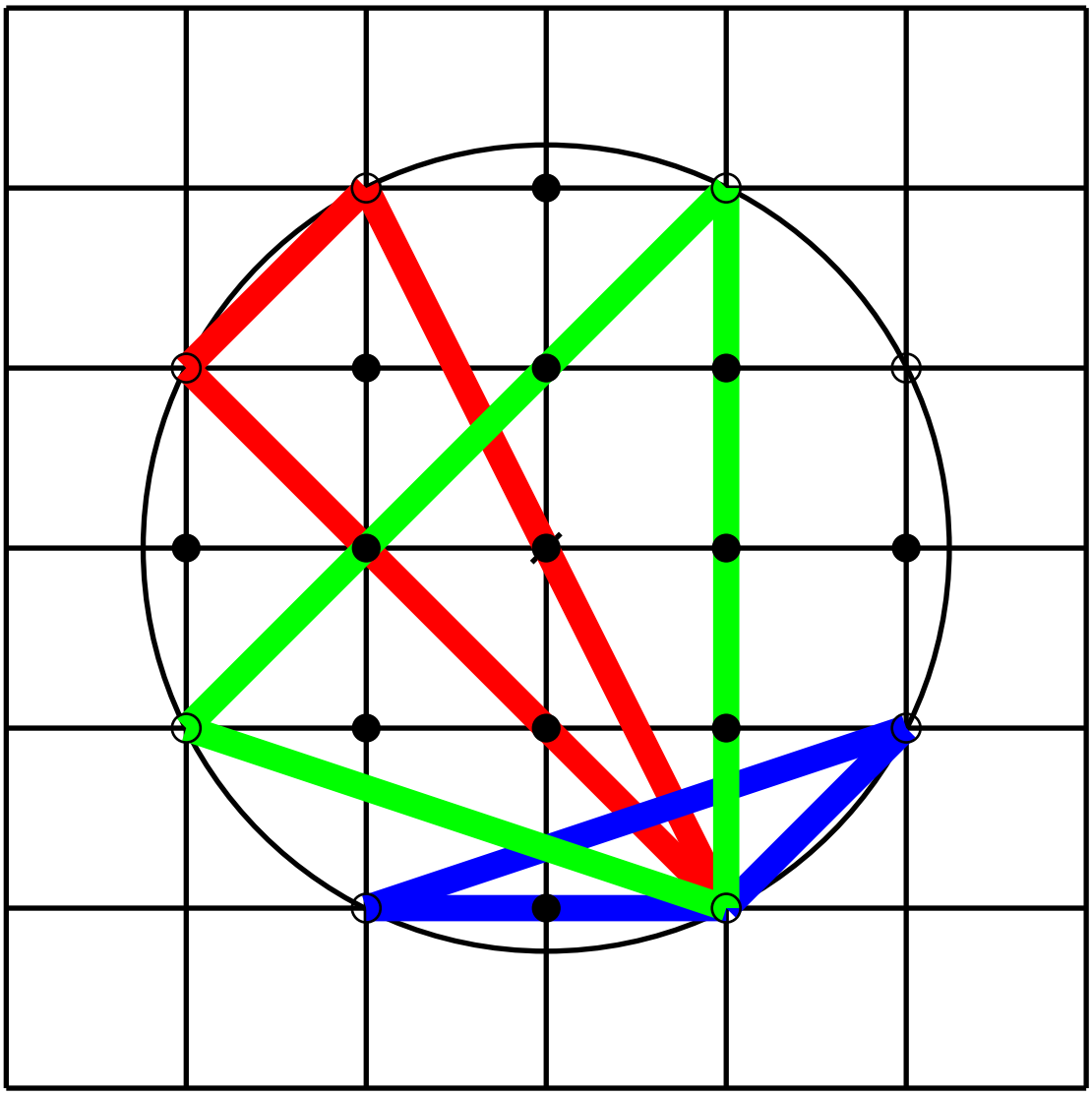
R^2 = 169 / 36 = 4.69444
R = 2.166667
X = 0 / 1
Y = 1 / 6
14 + 3 = 17

A192493(23) = 85, A192494(23) = 18
Triangles: O A



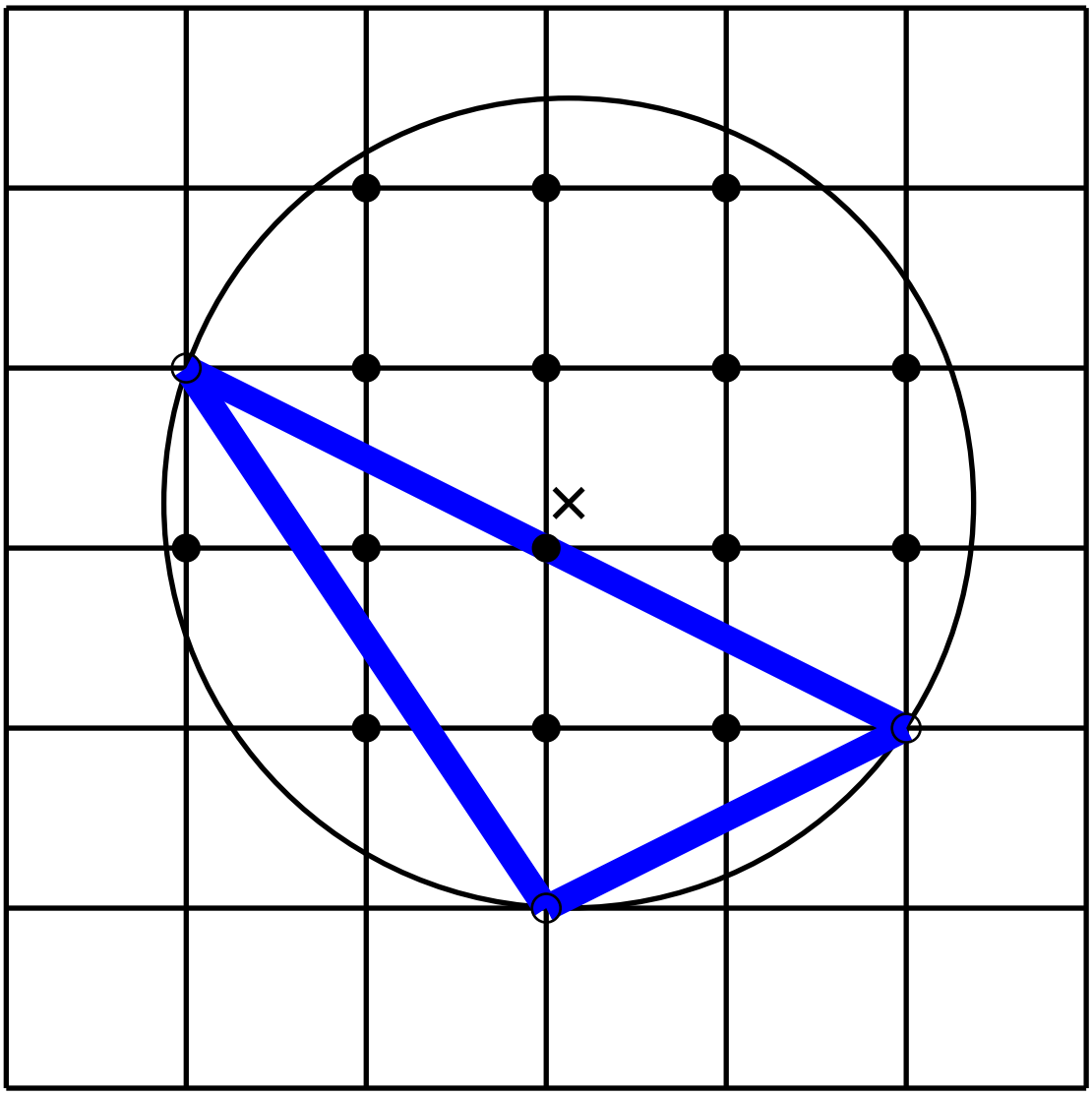
R^2 = 85/ 18 = 4.72222
R = 2.173067
X = 1/ 6
Y = 1/ 6
13 + 4 = 17

A192493(24) = 5, A192494(24) = 1
Triangles: O R A



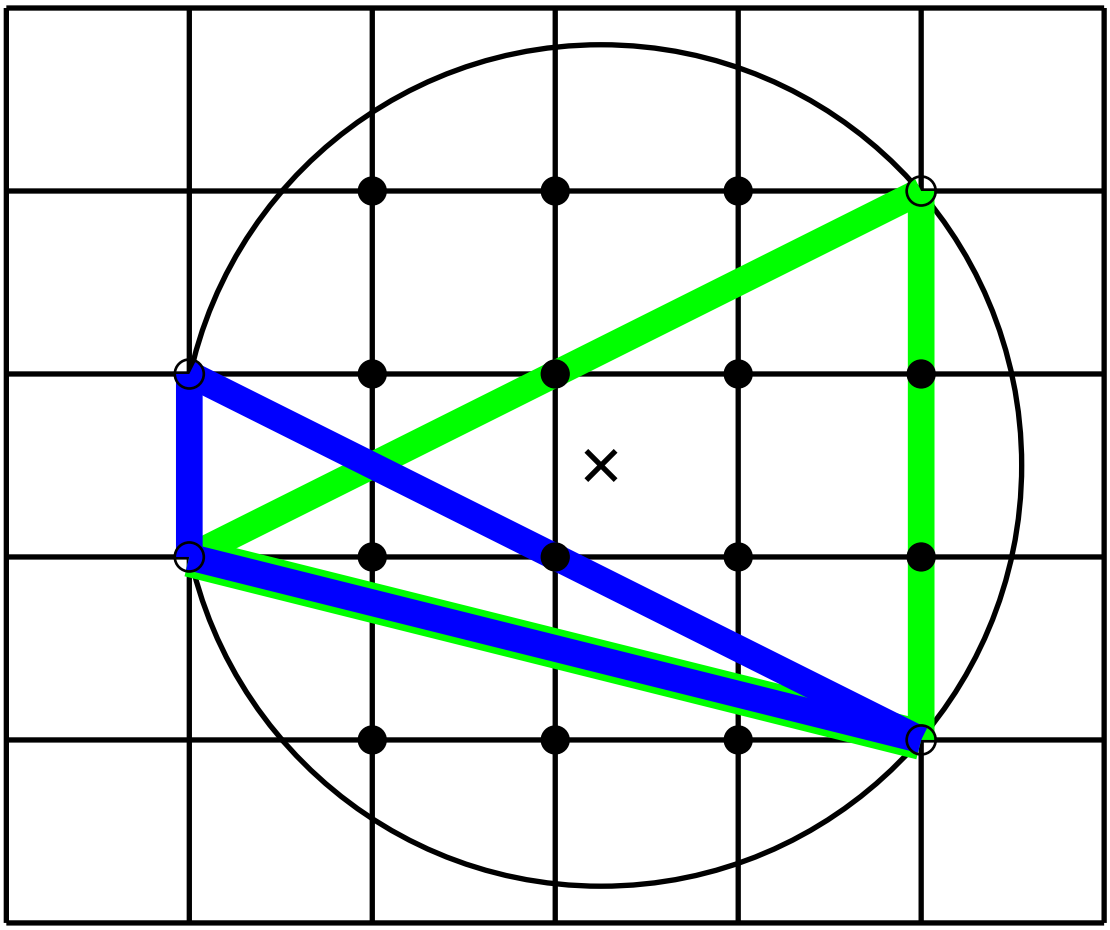
$R^2 = 5/1 = 5.00000$
R = 2.236068
X = 0/1
Y = 0/1
13 + 8 = 21

A192493(25) = 325, A192494(25) = 64
Triangles: O



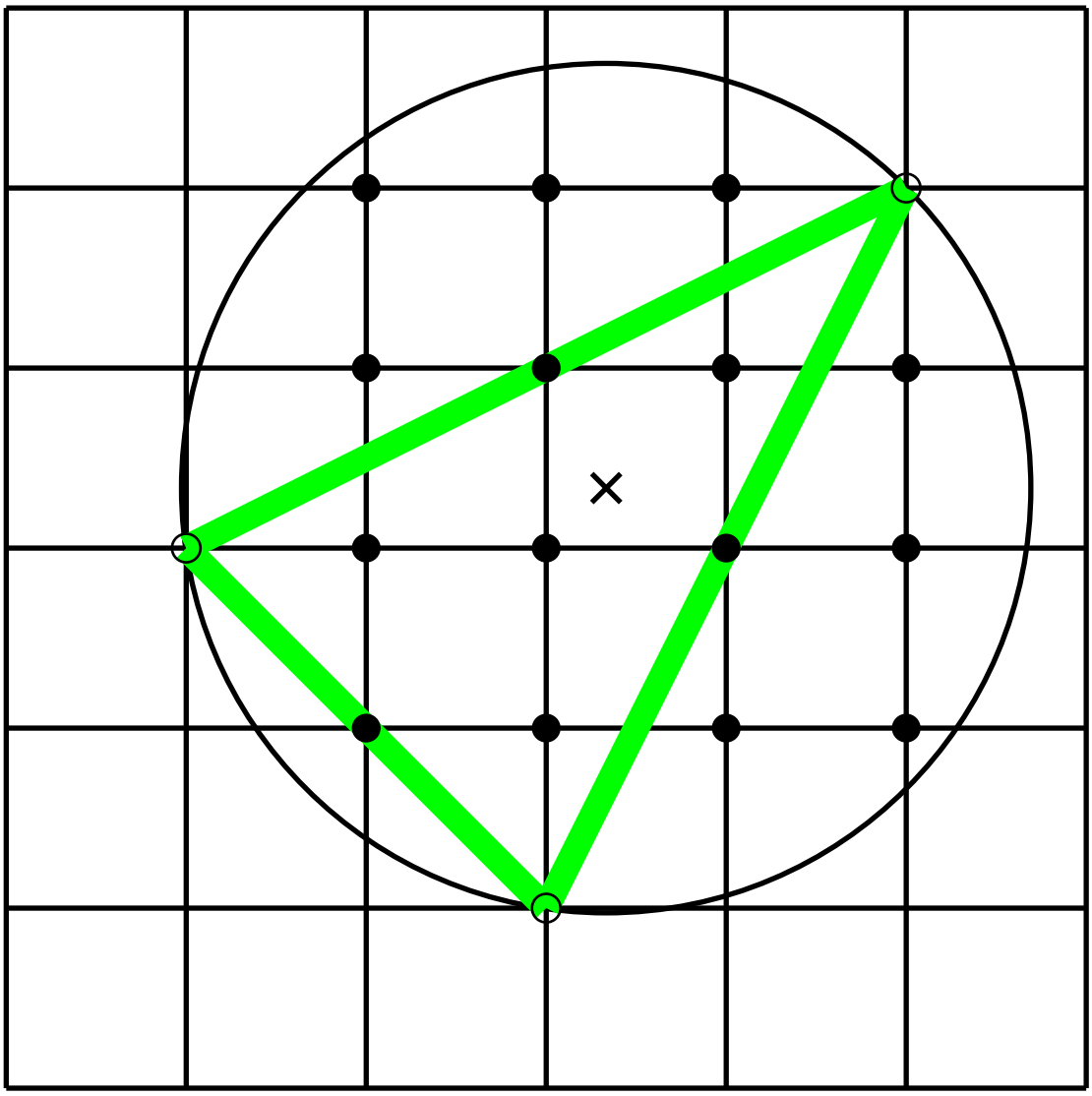
R^2 = 325/ 64 = 5.07812
R = 2.253470
X = 1/8
Y = 1/4
15 + 3 = 18

A192493(26) = 85, A192494(26) = 16
Triangles: O A



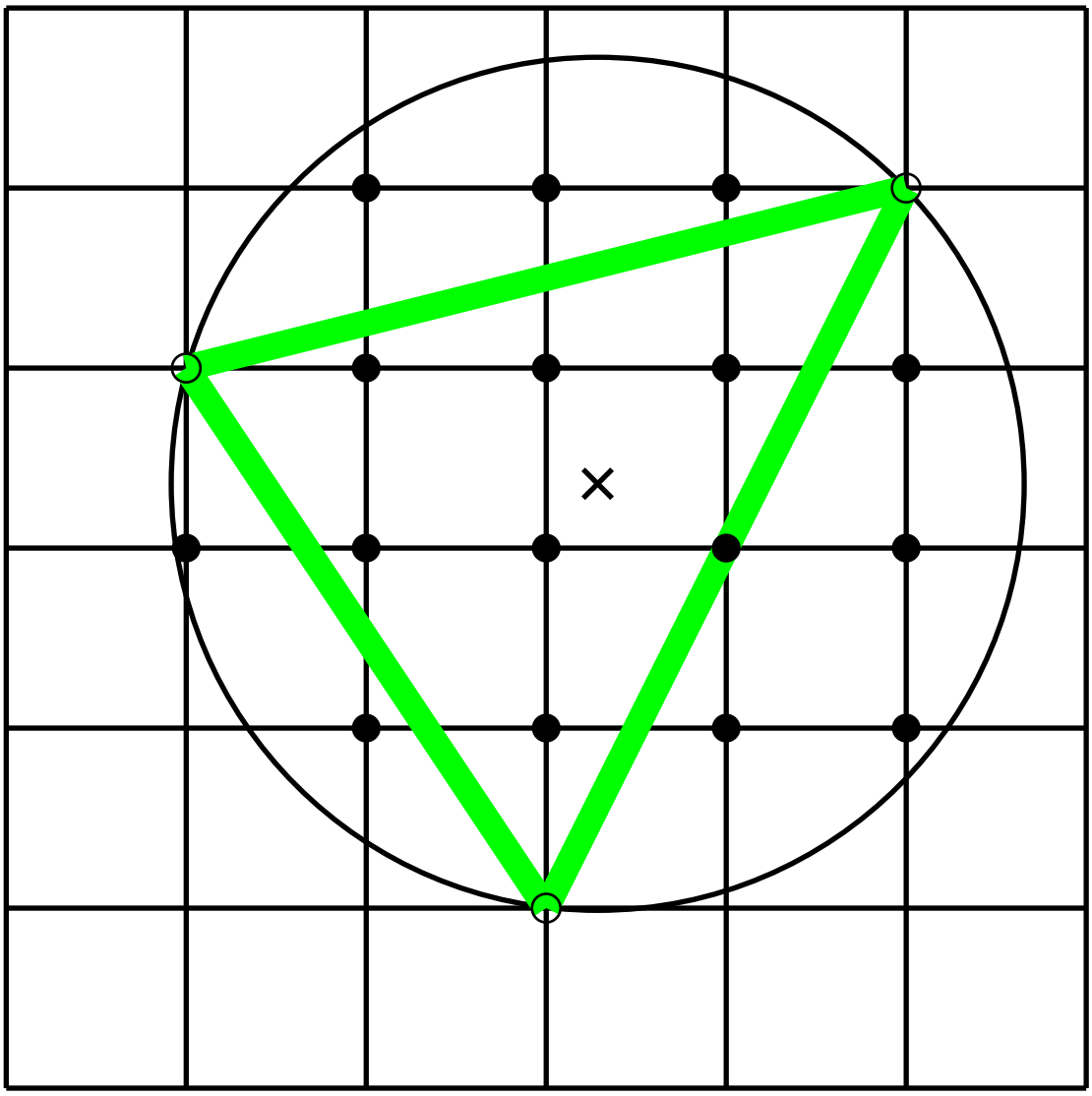
R^2 = 85/ 16 = 5.31250
R = 2.304886
X = 1/ 4
Y = 1/ 2
14 + 4 = 18

A192493(27) = 50, A192494(27) = 9
Triangles: A



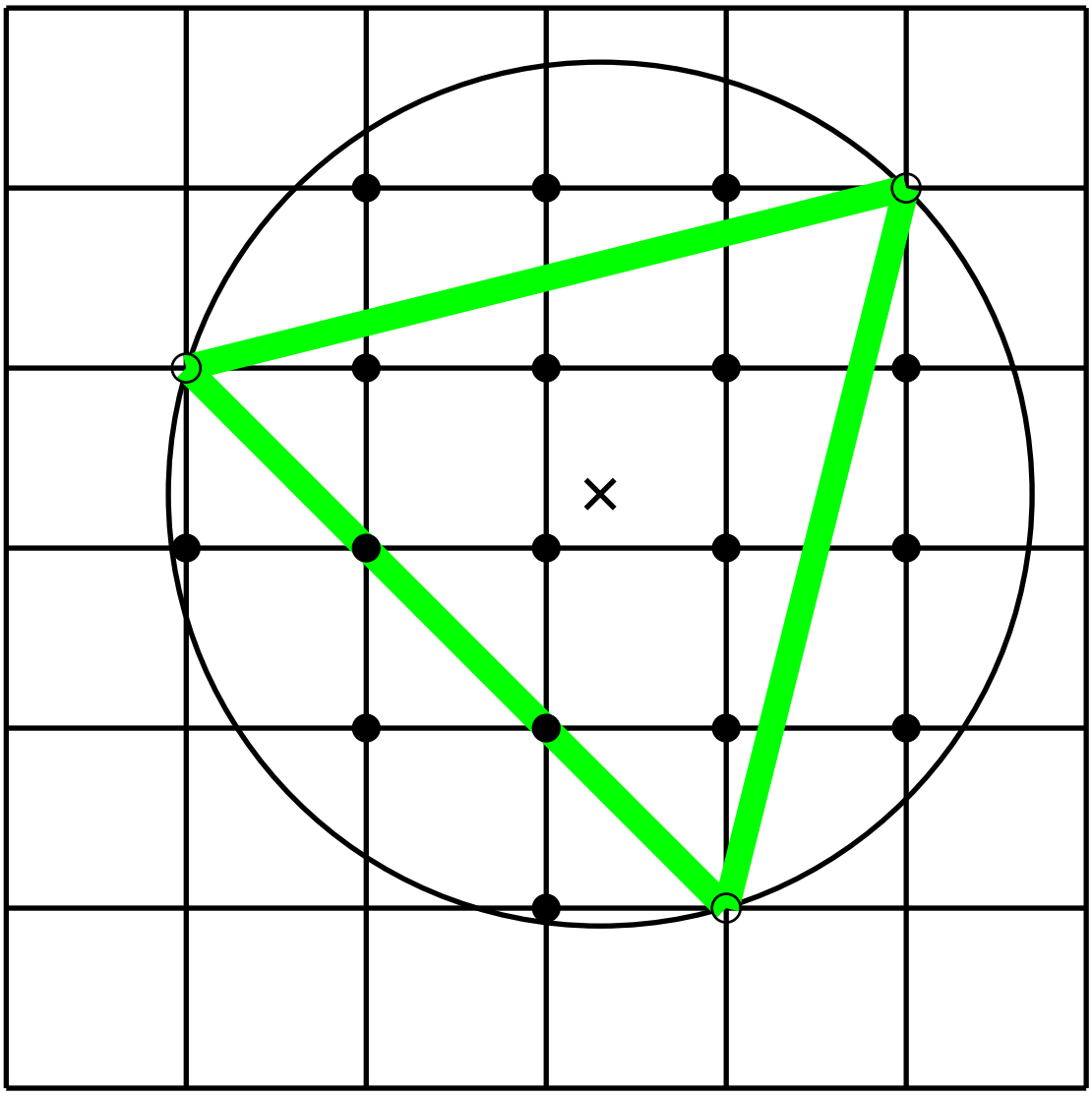
$R^2 = 50/9 = 5.555556$
$R = 2.357023$
$X = 1/3$
$Y = 1/3$
$15 + 3 = 18$

A192493(28) = 1105, A192494(28) = 196
Triangles: A



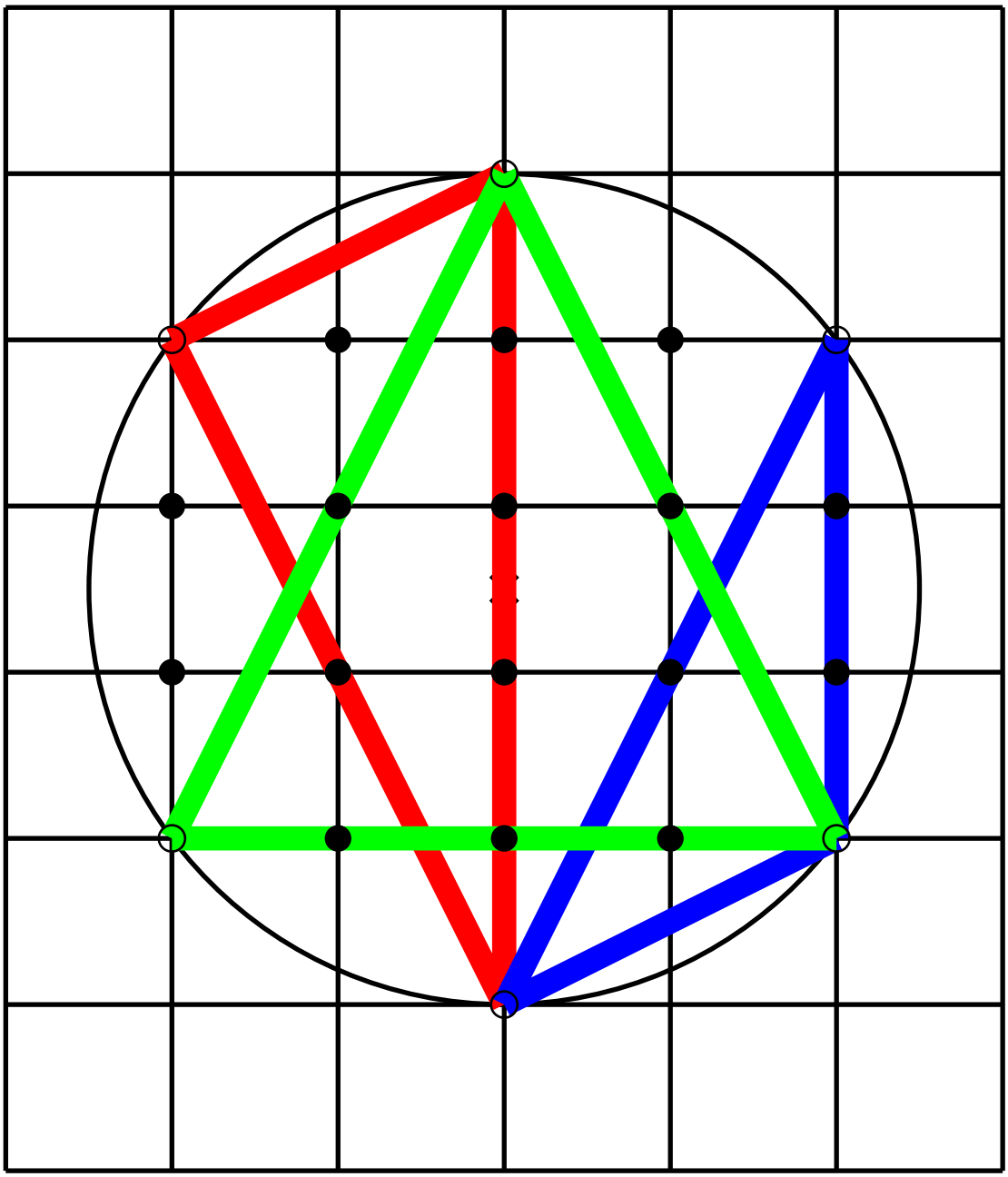
R^2 = 1105/ 196 = 5.63776
R = 2.374396
X = 2/ 7
Y = 5/ 14
16 + 3 = 19

A192493(29) = 289, A192494(29) = 50
Triangles: A



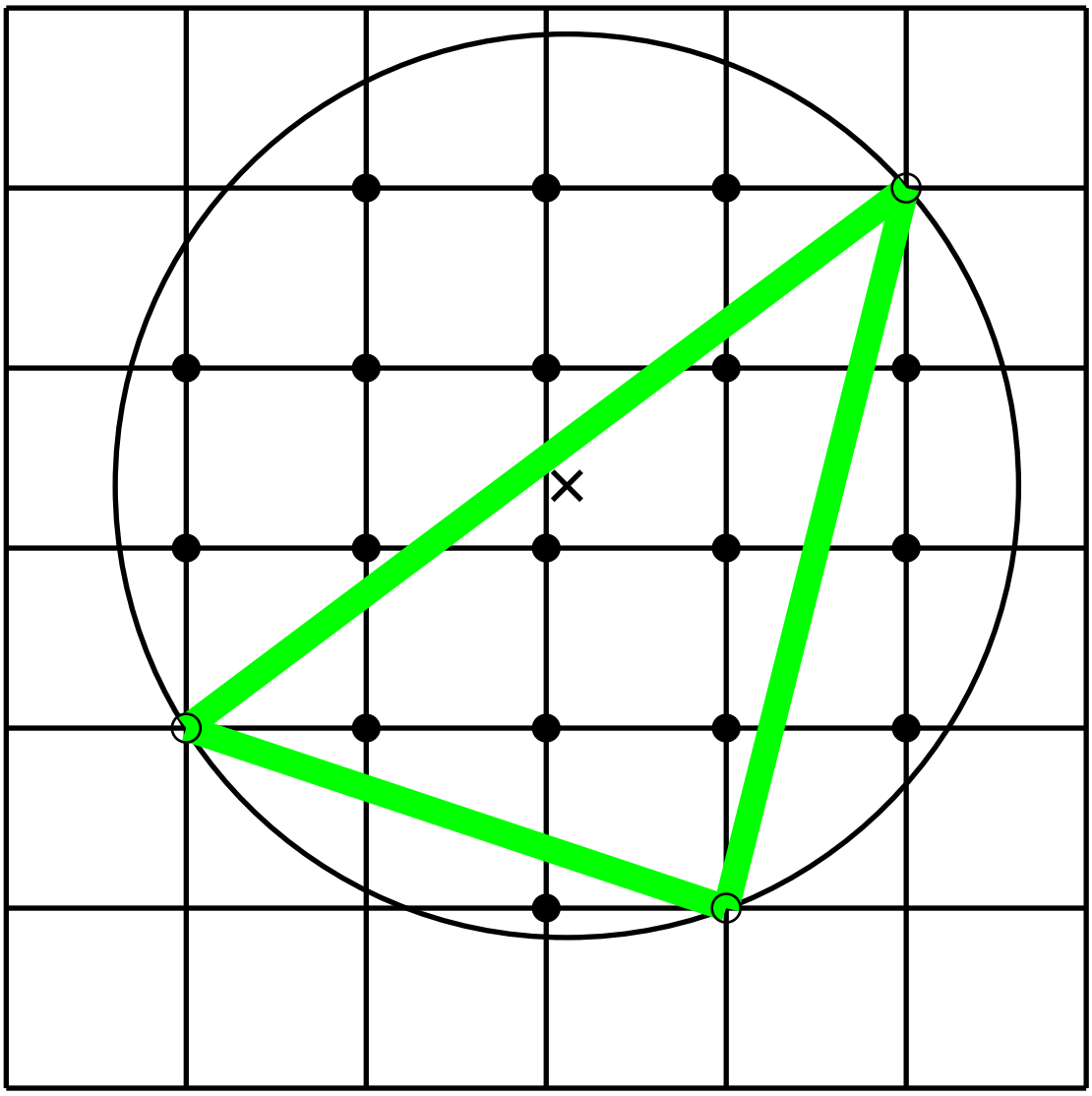
R^2 = 289/ 50 = 5.78000
R = 2.404163
X = 3/ 10
Y = 3/ 10
17 + 3 = 20

A192493(30) = 25, A192494(30) = 4
Triangles: O R A



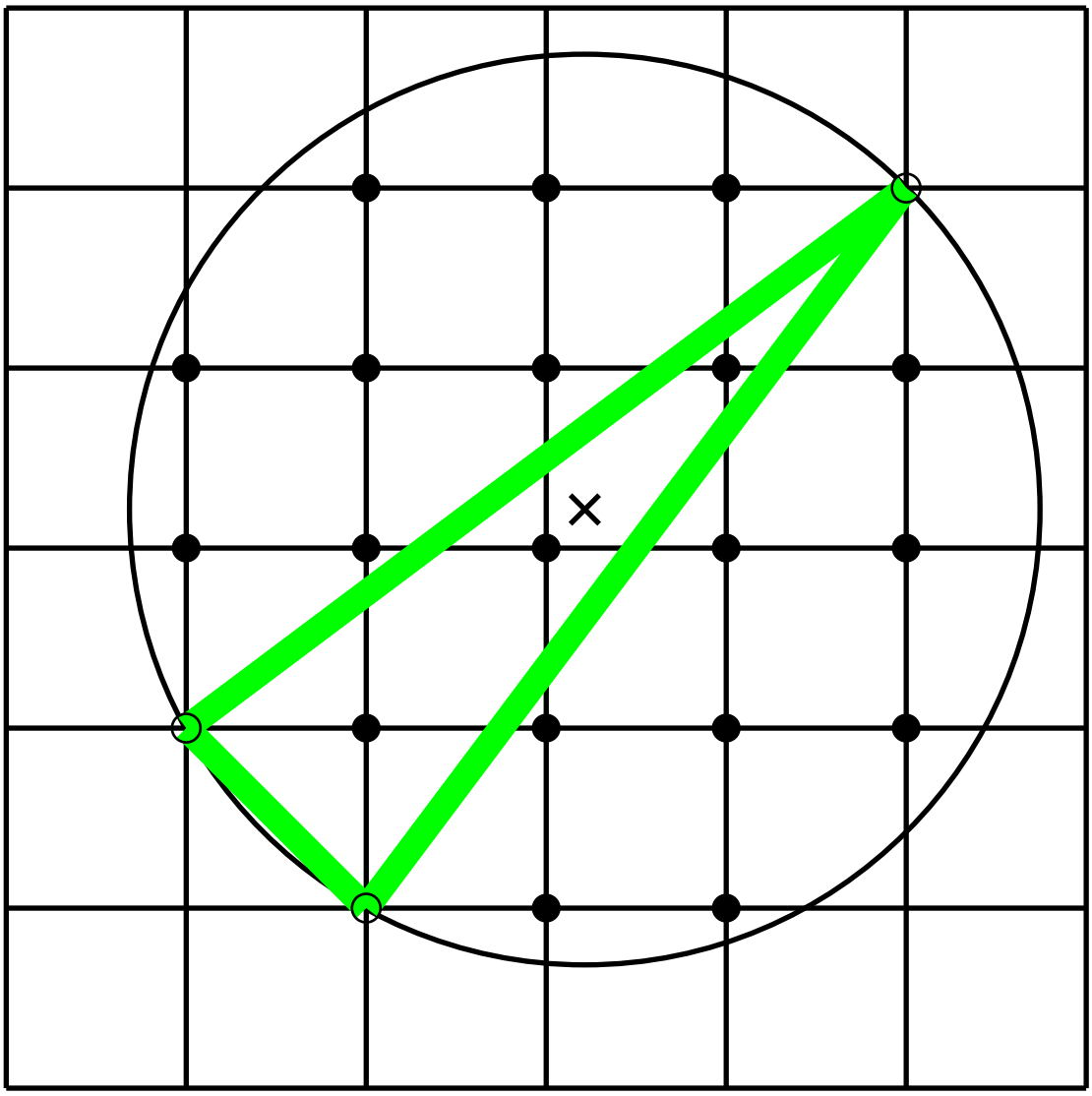
$R^2 = 25/4 = 6.25000$
$R = 2.500000$
$X = 0/1$
$Y = 1/2$
$16 + 6 = 22$

A192493(31) = 2125, A192494(31) = 338
Triangles: A



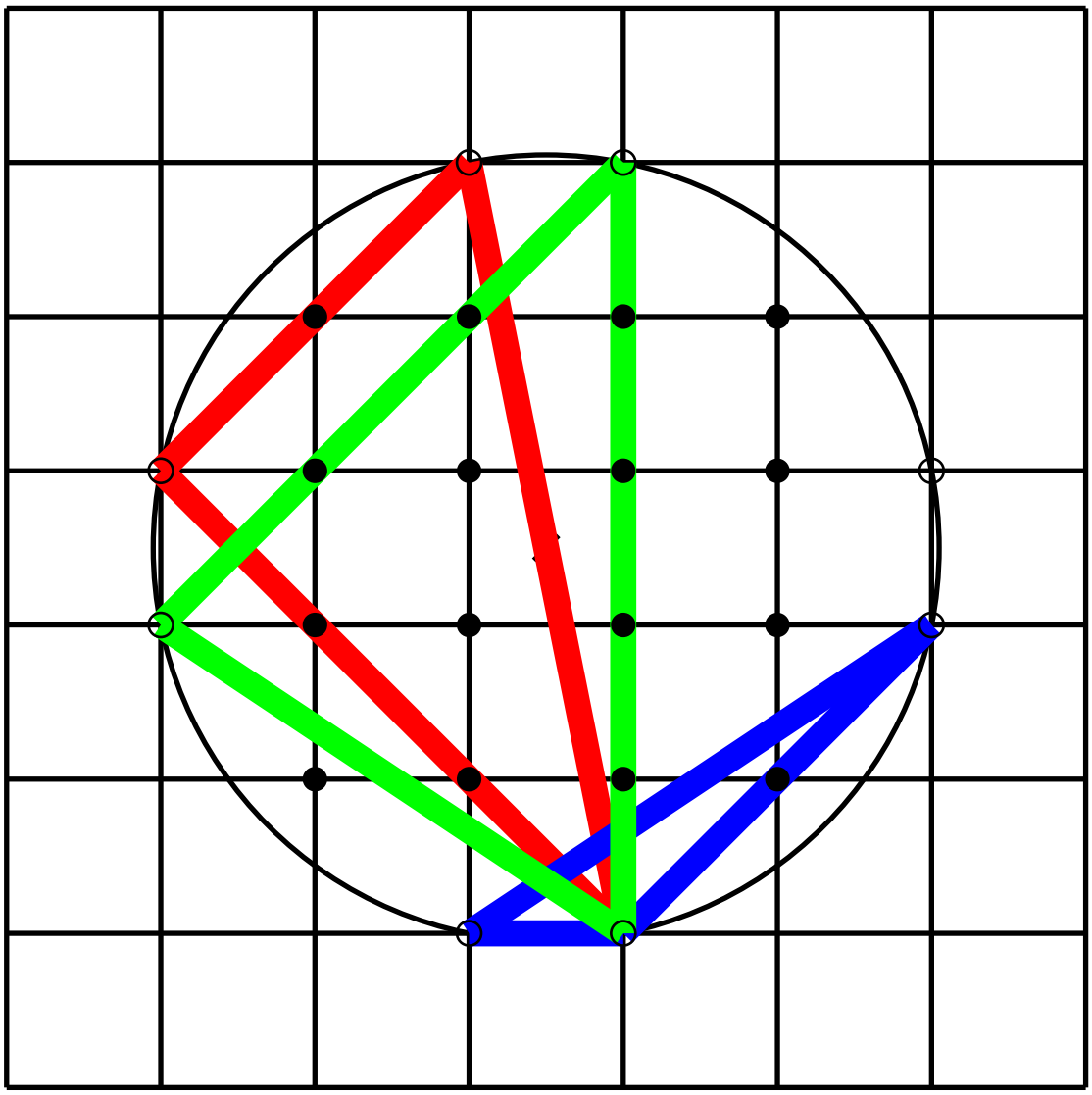
R^2 = 2125/ 338 = 6.28698
R = 2.507386
X = 3/ 26
Y = 9/ 26
18 + 3 = 21

A192493(32) = 625, A192494(32) = 98
Triangles: A



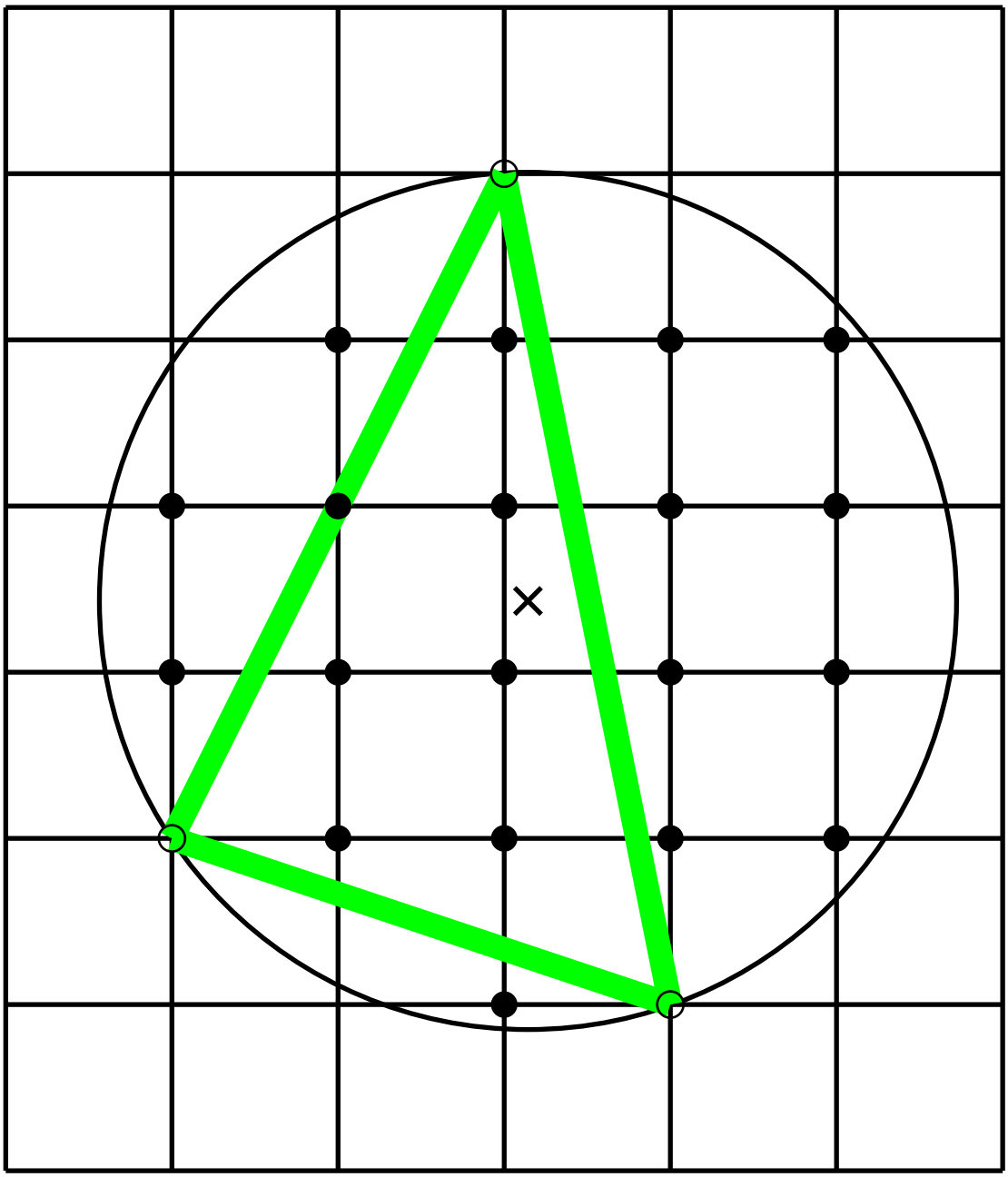
R^2 = 625/ 98 = 6.377551
R = 2.525381
X = 3/ 14
Y = 3/ 14
19 + 3 = 22

A192493(33) = 13, A192494(33) = 2
Triangles: O R A



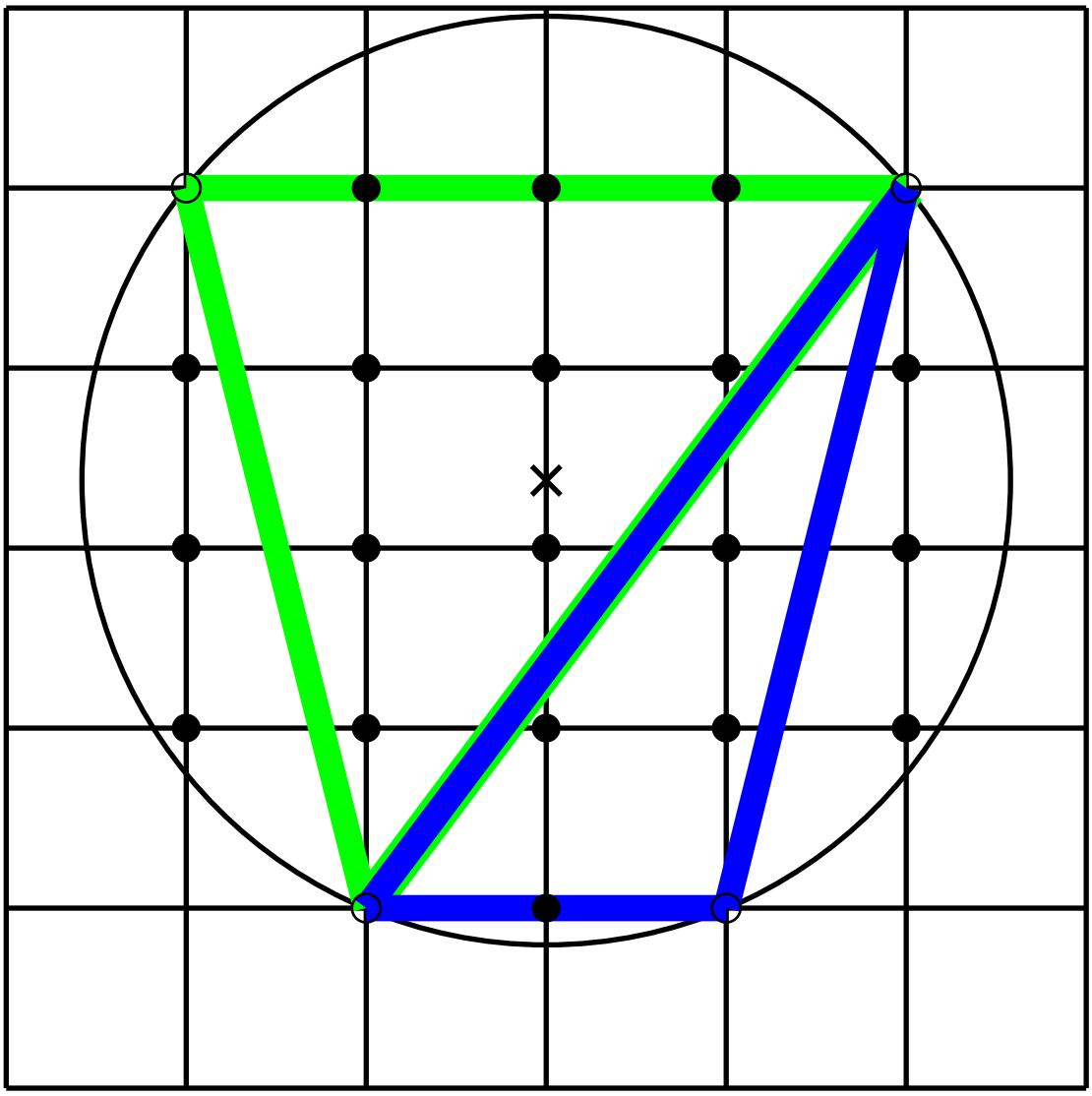
$R^2 = 13/2 = 6.50000$
$R = 2.549510$
$X = 1/2$
$Y = 1/2$
$16 + 8 = 24$

A192493(34) = 325, A192494(34) = 49
Triangles: A



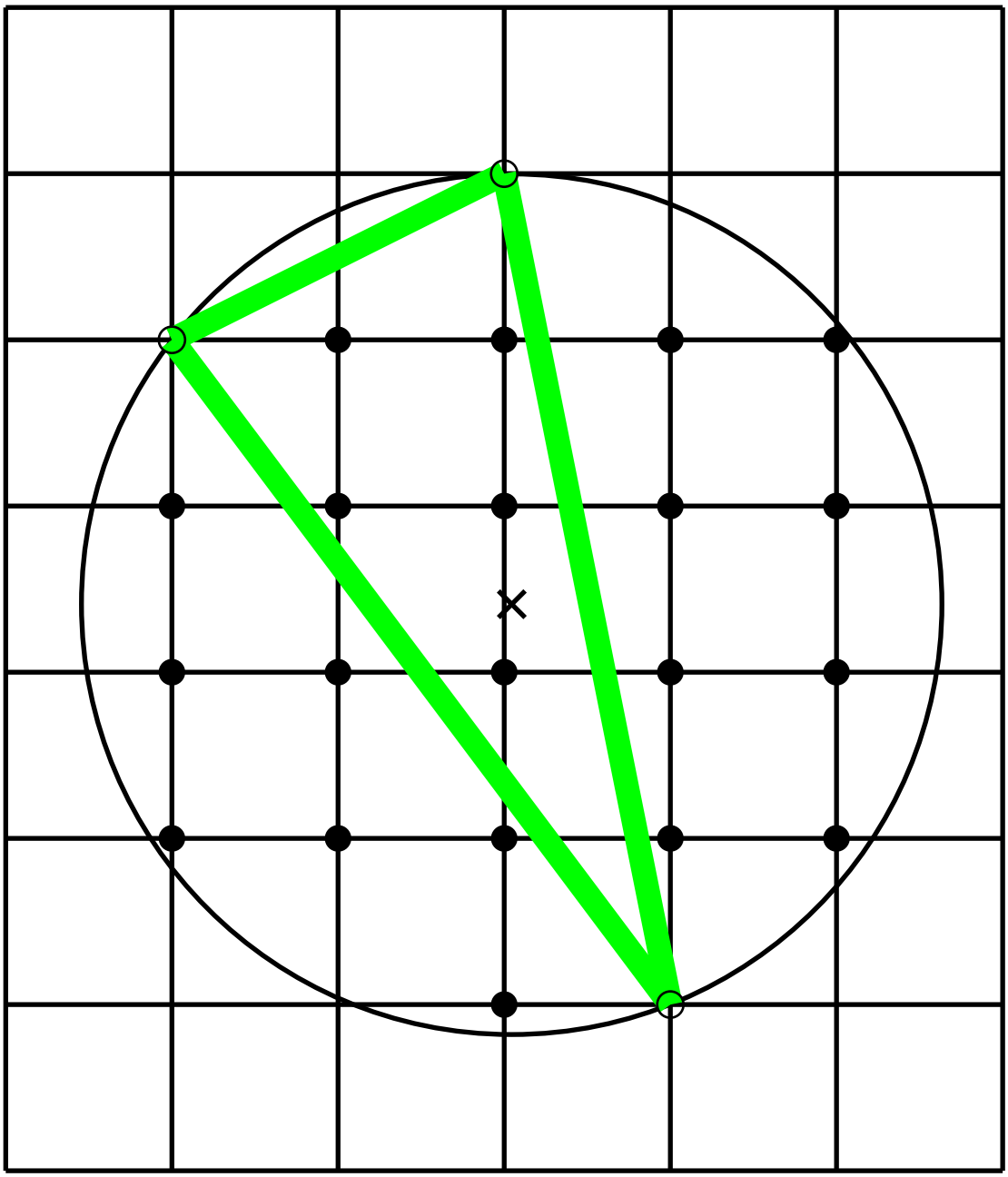
R^2 = 325/49 = 6.63265
R = 2.575394
X = 1/7
Y = 3/7
19 + 3 = 22

A192493(35) = 425, A192494(35) = 64
Triangles: O A



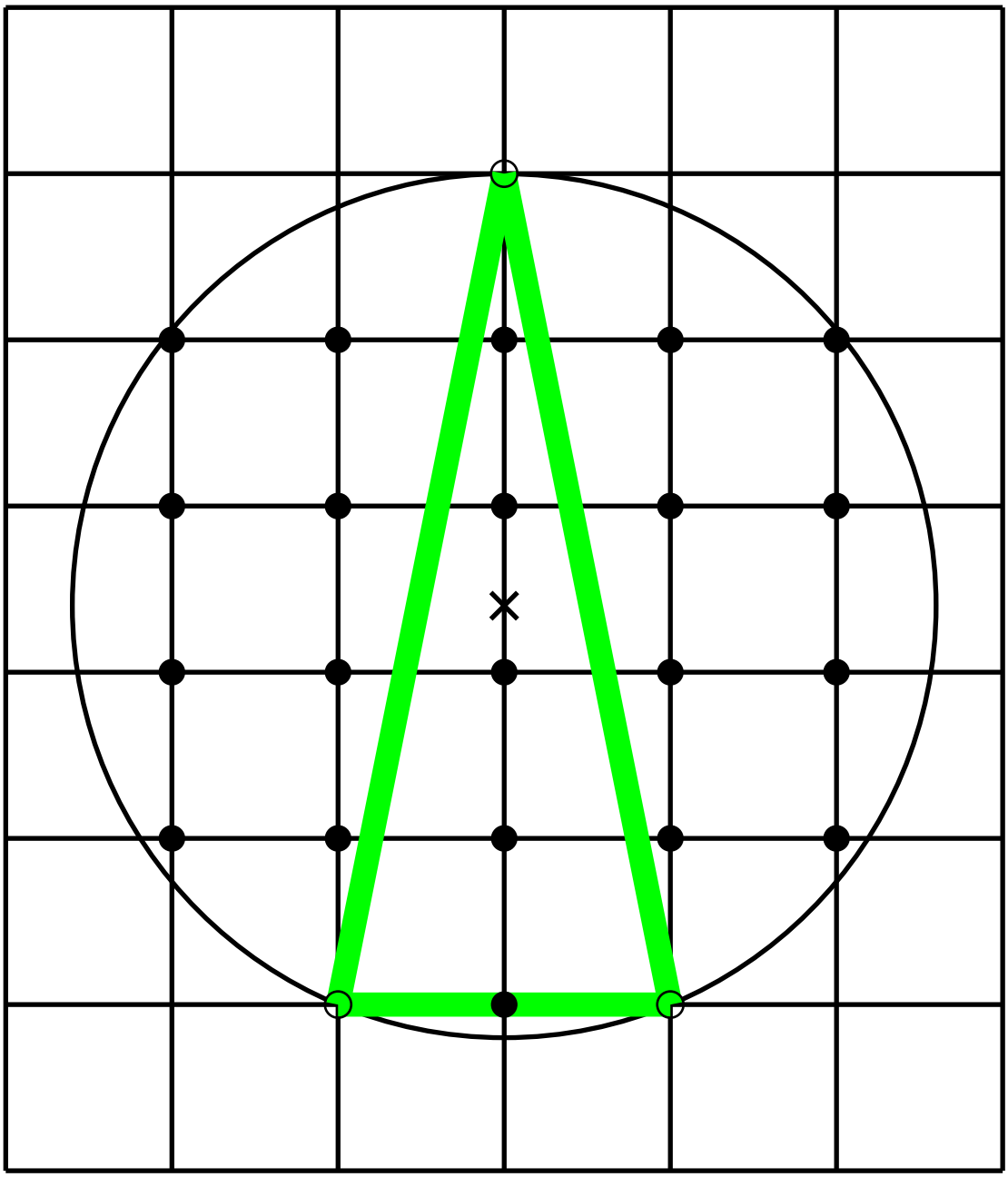
R^2 = 425/ 64 = 6.64062
R = 2.576941
X = 0/ 1
Y = 3/ 8
19 + 4 = 23

A192493(36) = 1625, A192494(36) = 242
Triangles: A



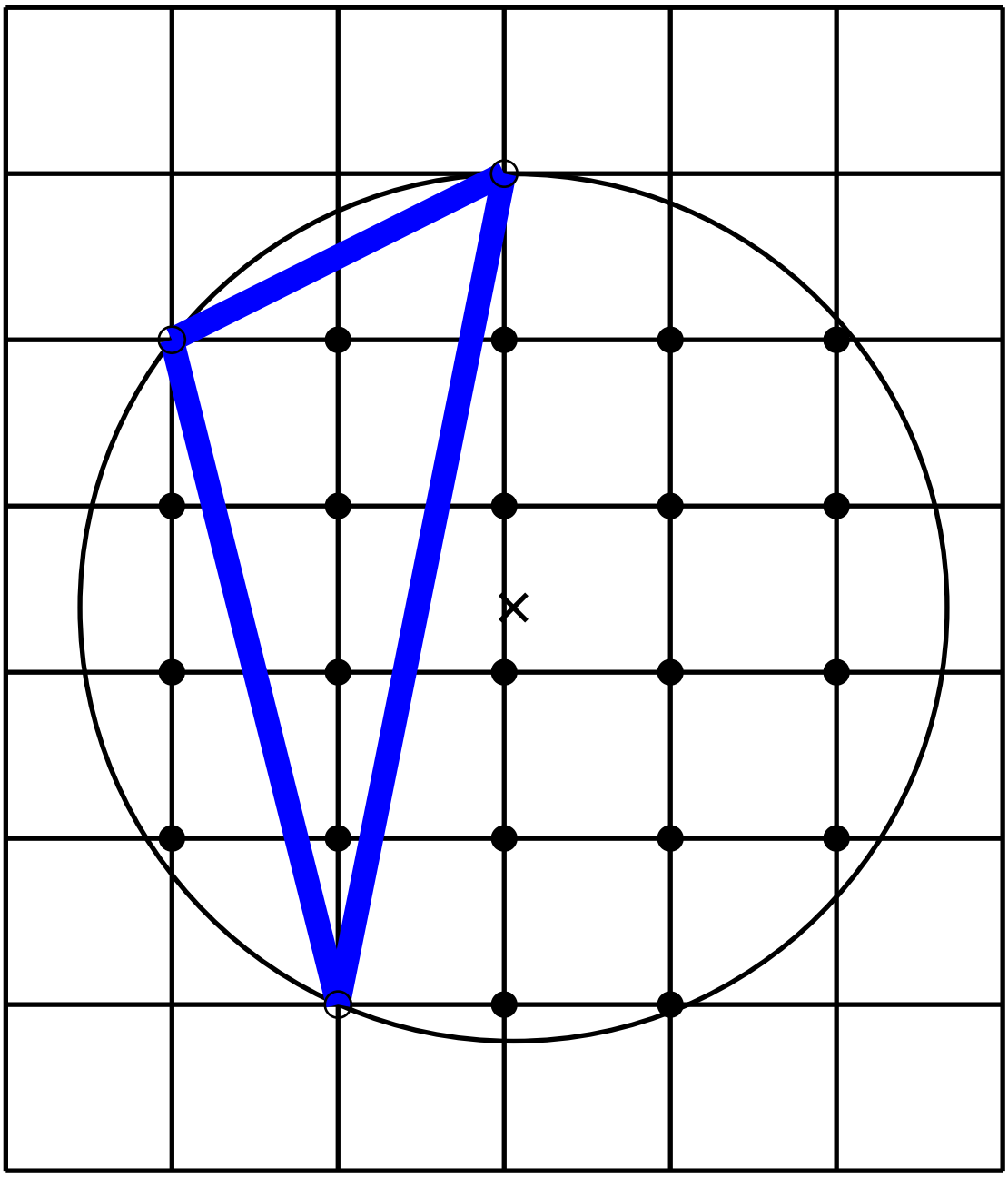
R^2 = 1625/242 = 6.71488
R = 2.591308
X = 1/22
Y = 9/22
20 + 3 = 23

A192493(37) = 169, A192494(37) = 25
Triangles: A



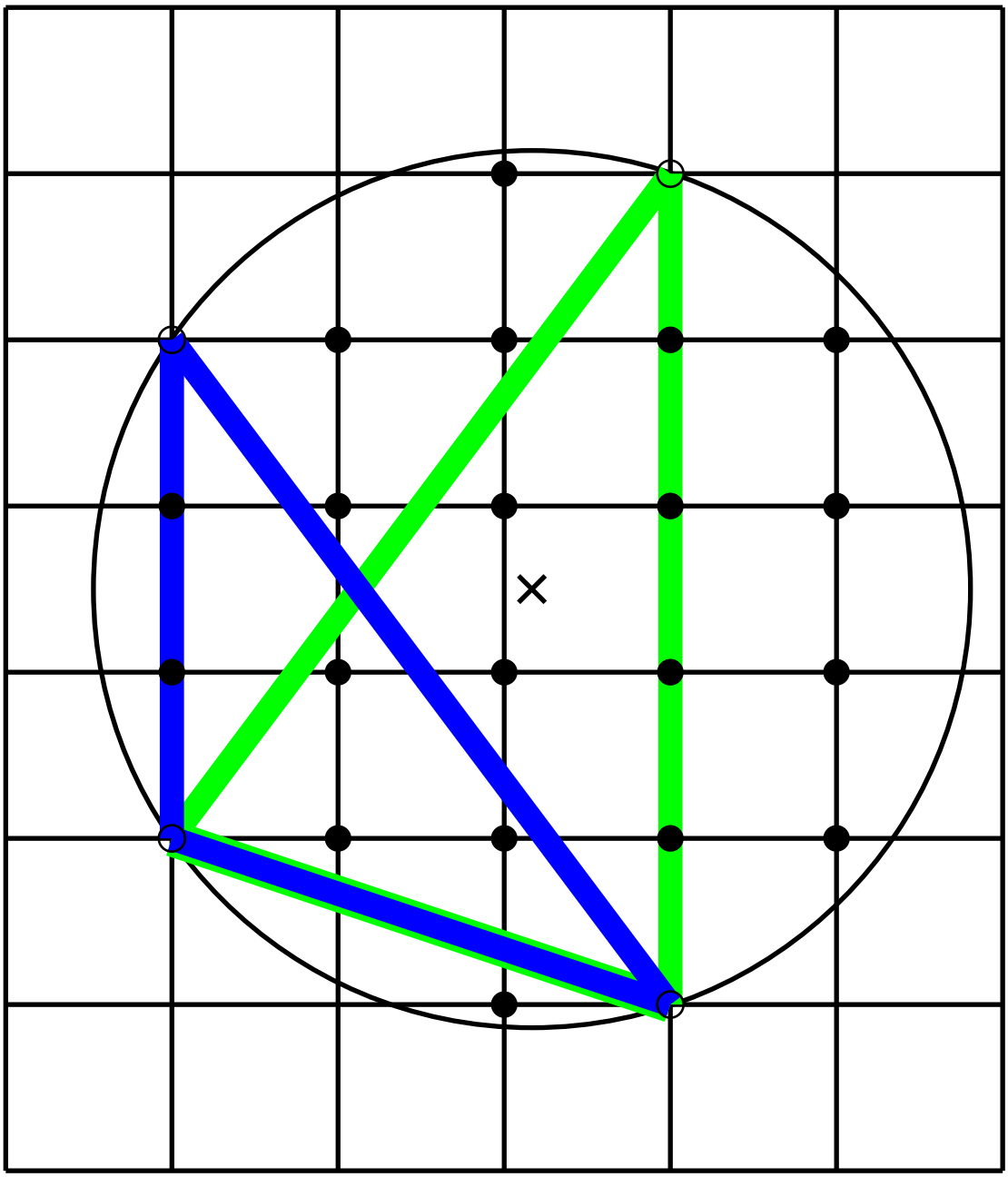
R^2 = 169/ 25 = 6.76000
R = 2.600000
X = 0 / 1
Y = 2 / 5
21 + 3 = 24

A192493(38) = 1105, A192494(38) = 162
Triangles: O



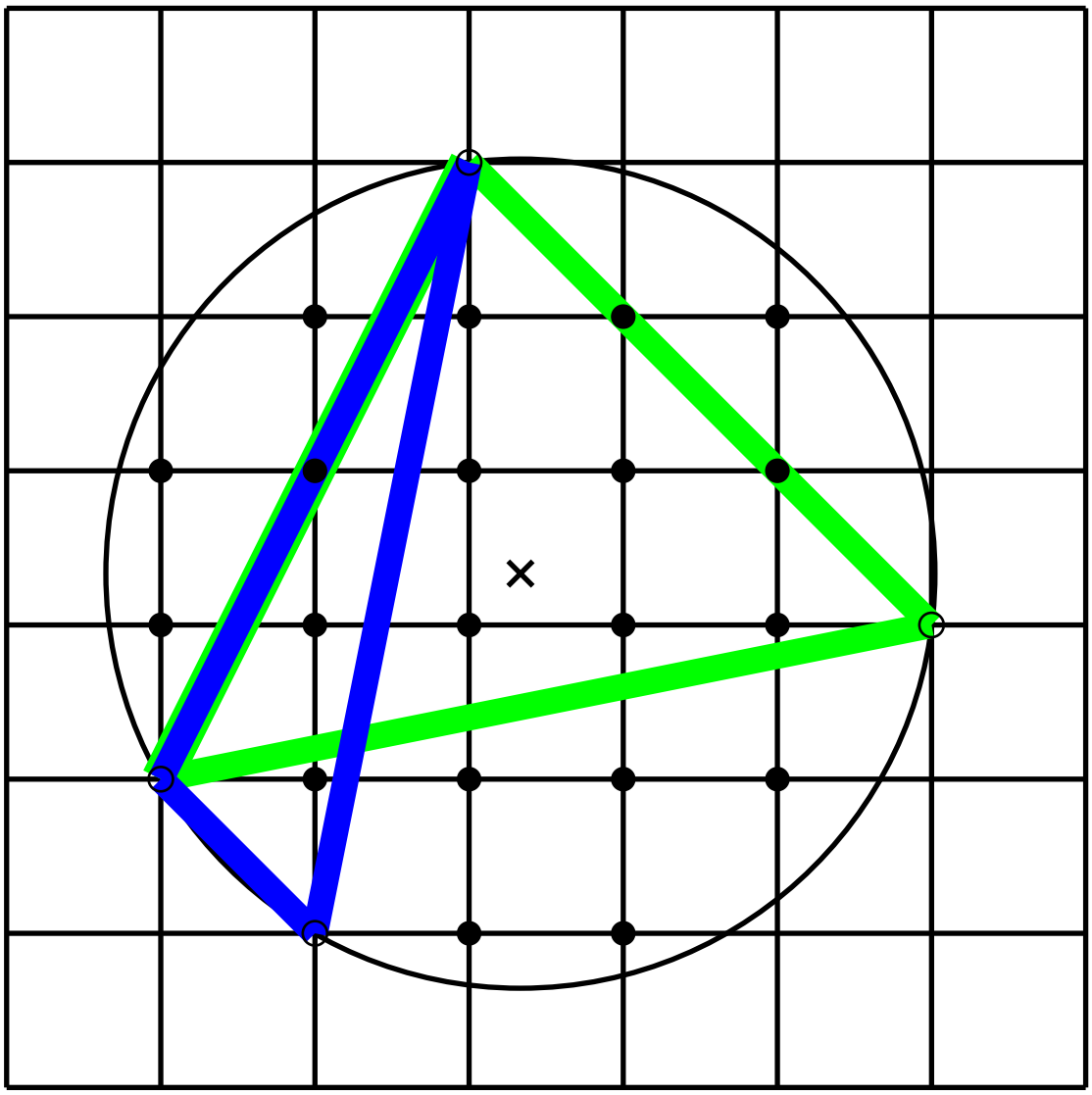
R^2 = 1105/ 162 = 6.82099
R = 2.611702
X = 1/ 18
Y = 7/ 18
21 + 3 = 24

A192493(39) = 125, A192494(39) = 18
Triangles: O A



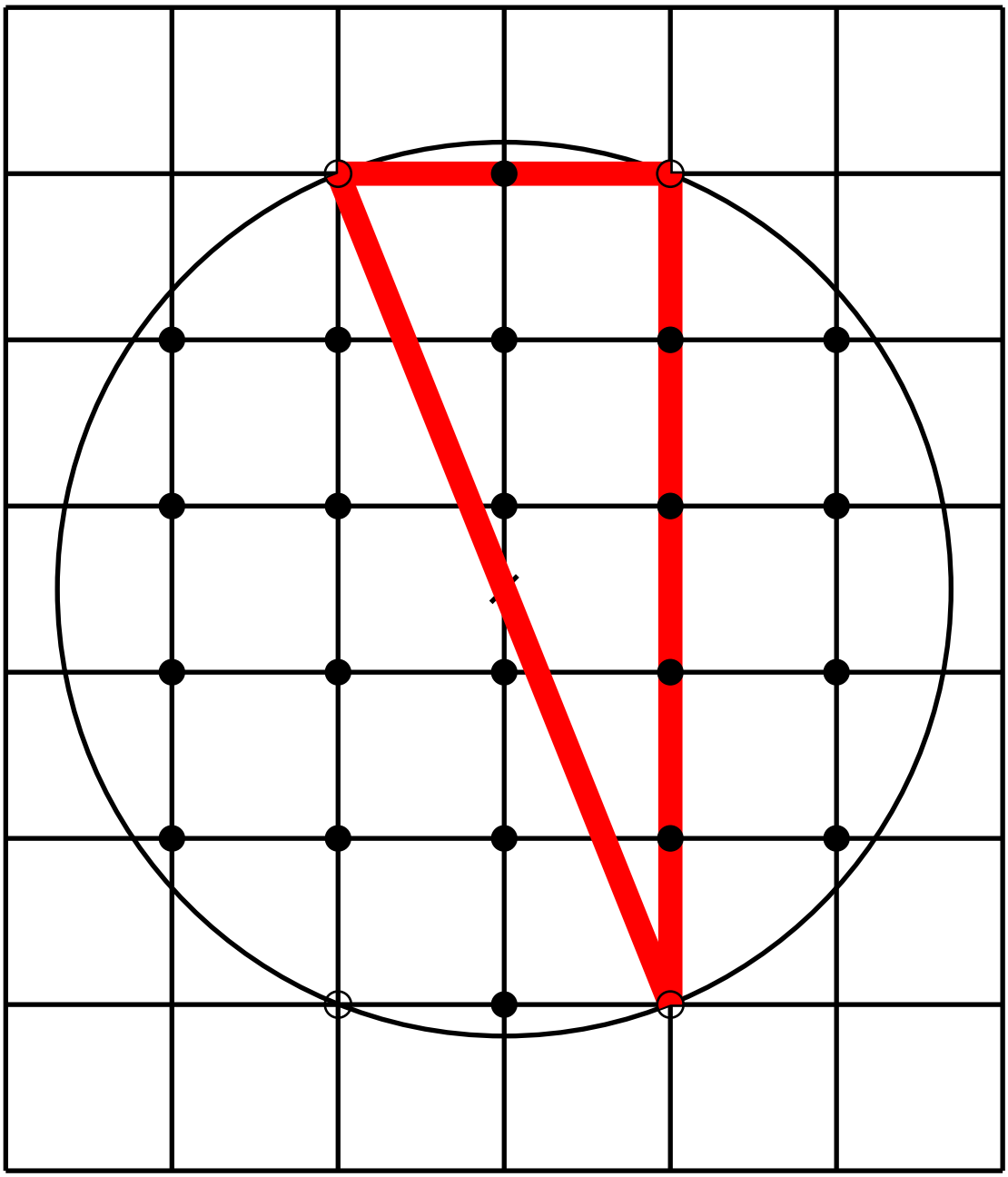
R^2 = 125/18 = 6.94444
R = 2.635231
X = 1/6
Y = 1/2
20 + 4 = 24

A192493(40) = 65, A192494(40) = 9
Triangles: O A



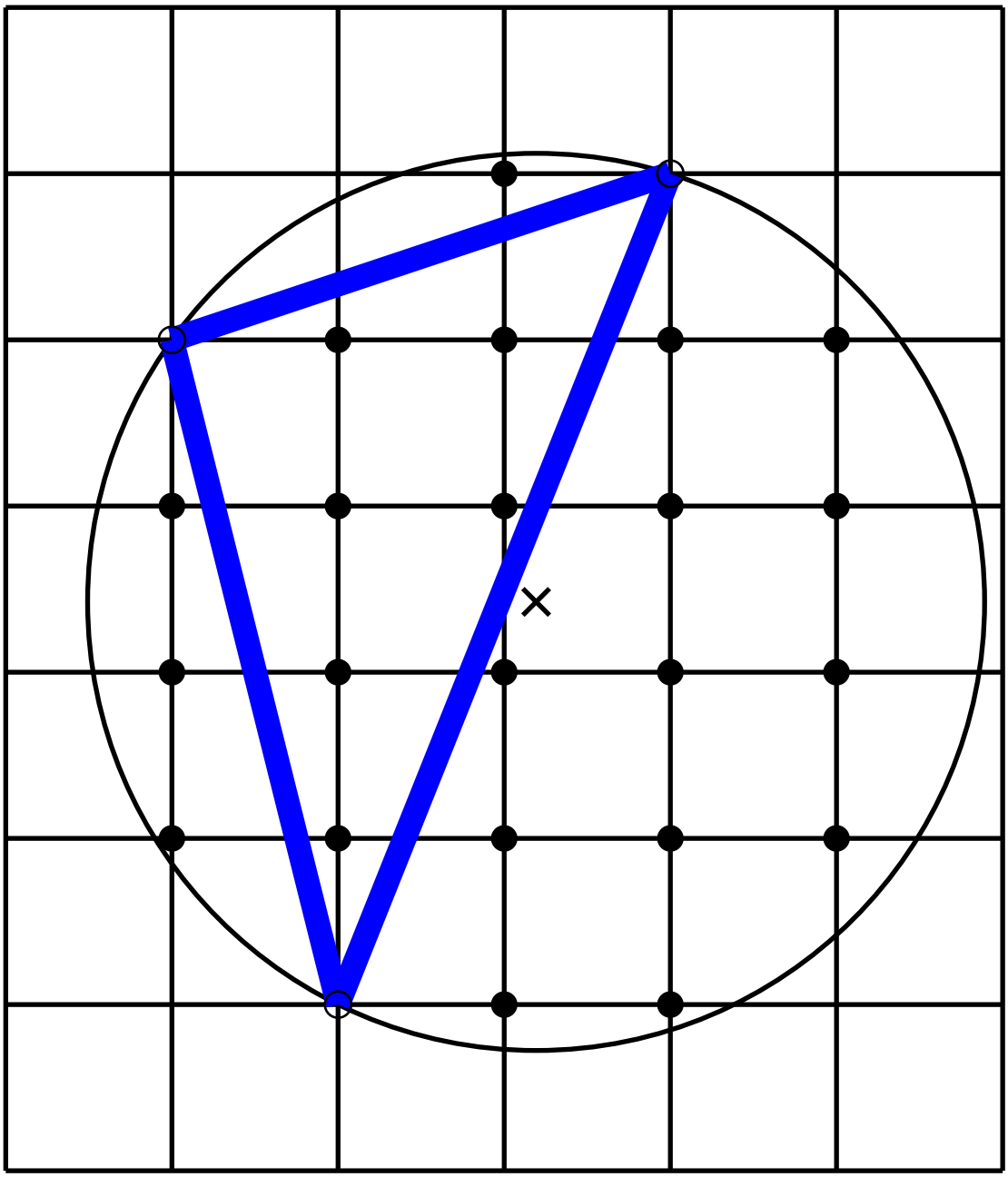
$R^2 = 65/9 = 7.222222$
$R = 2.687419$
$X = 1/3$
$Y = 1/3$
$20 + 4 = 24$

A192493(41) = 29, A192494(41) = 4
Triangles: R



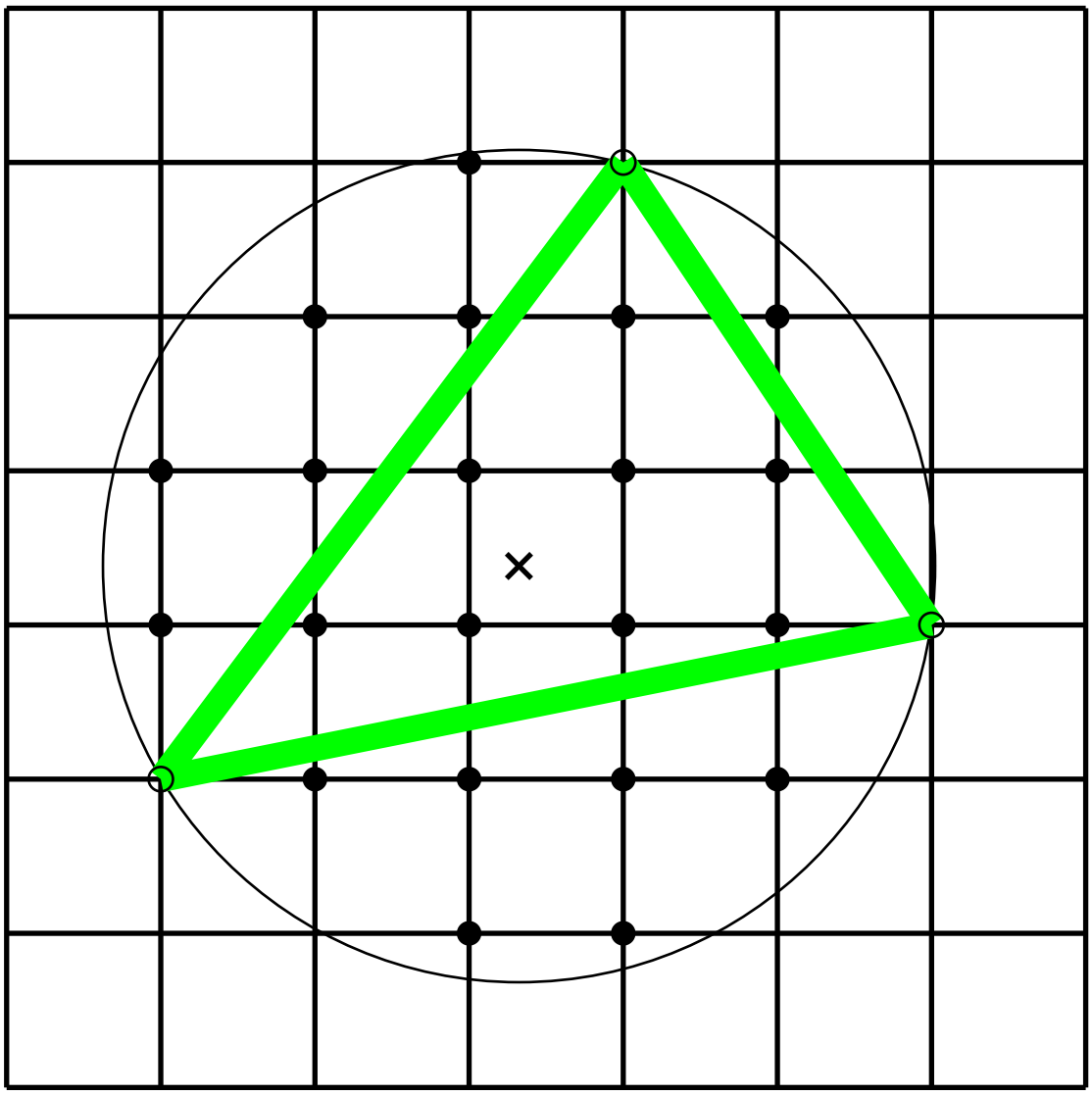
R^2 = 29/4 = 7.25000
R = 2.692582
X = 0/1
Y = 1/2
22 + 4 = 26

A192493(42) = 2465, A192494(42) = 338
Triangles: O



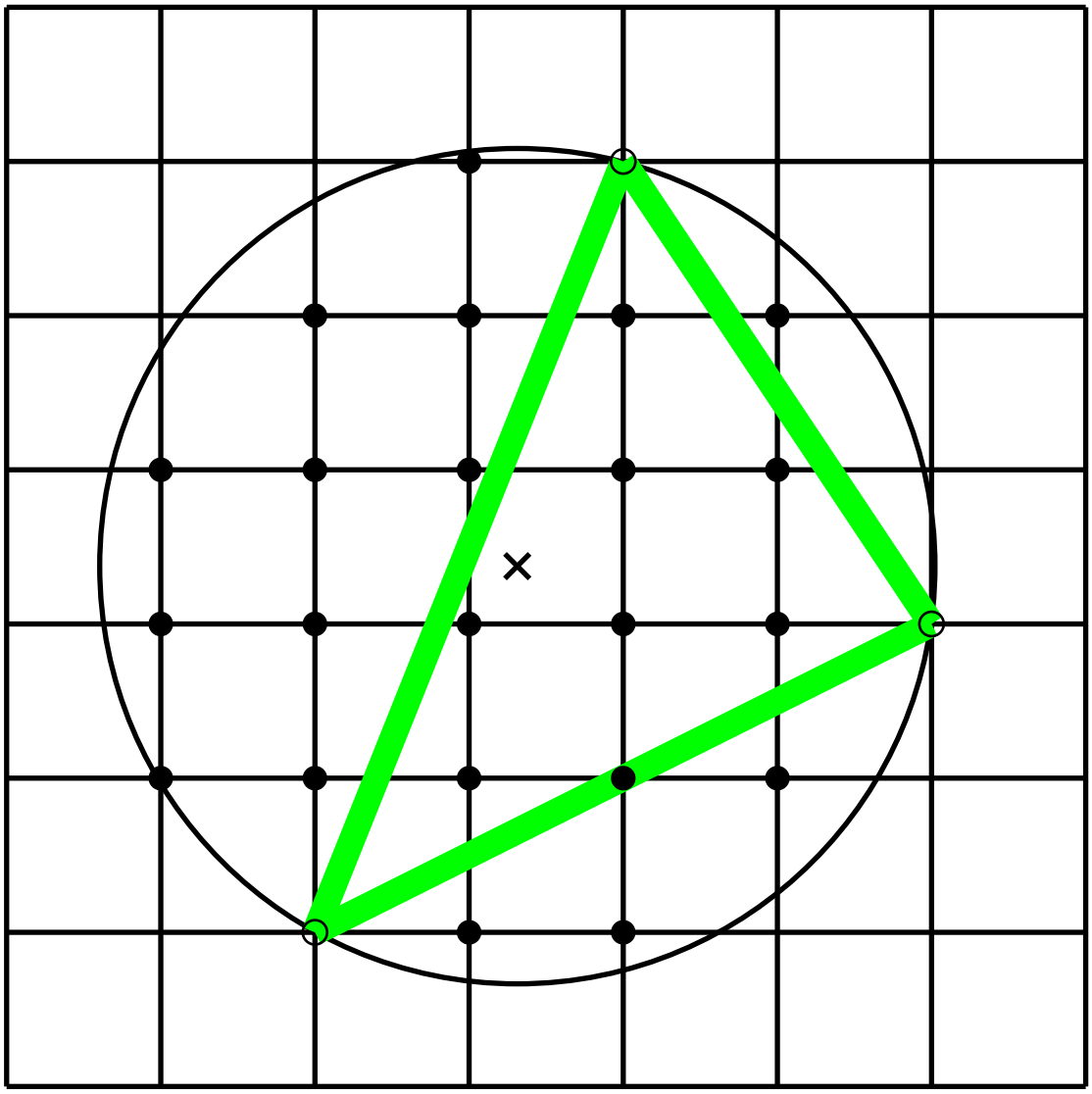
R^2 = 2465/338 = 7.29290
R = 2.700537
X = 5/26
Y = 11/26
22 + 3 = 25

A192493(43) = 4225, A192494(43) = 578
Triangles: A



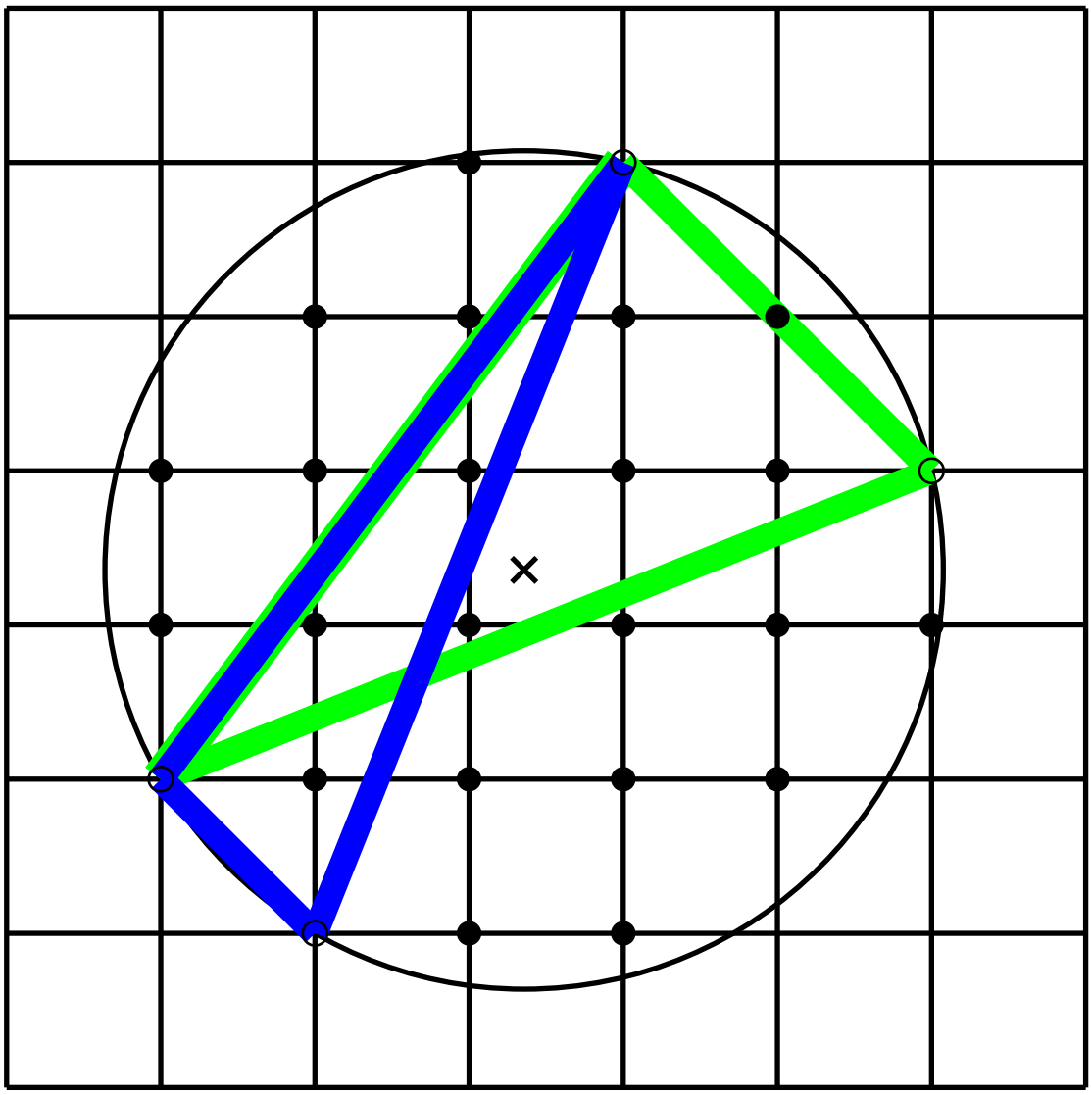
R^2 = 4225/578 = 7.30969
R = 2.703644
X = 11/34
Y = 13/34
21 + 3 = 24

A192493(44) = 1885, A192494(44) = 256
Triangles: A



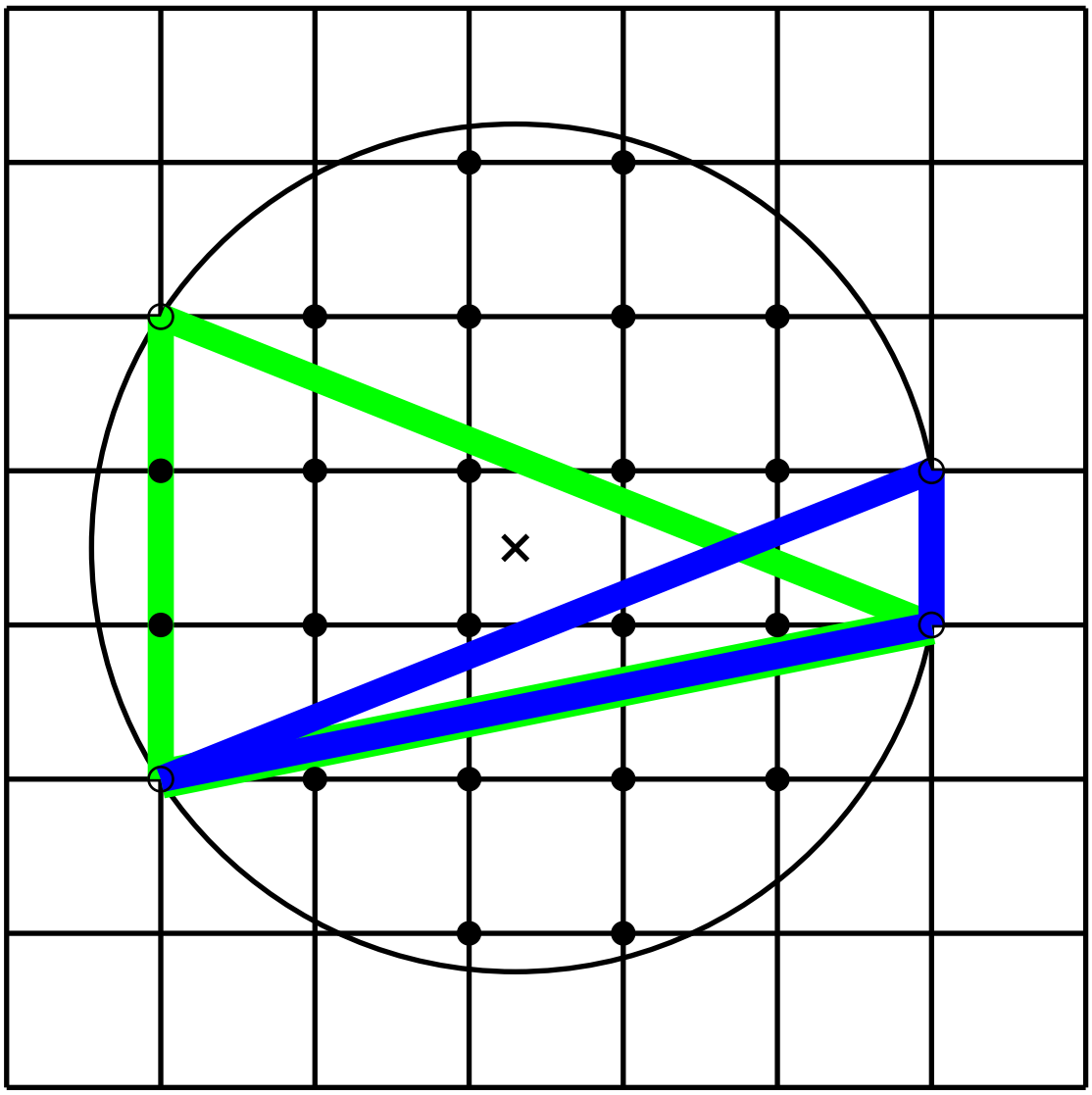
R^2 = 1885/256 = 7.36328
R = 2.713537
X = 5/16
Y = 3/8
22 + 3 = 25

A192493(45) = 725, A192494(45) = 98
Triangles: O A



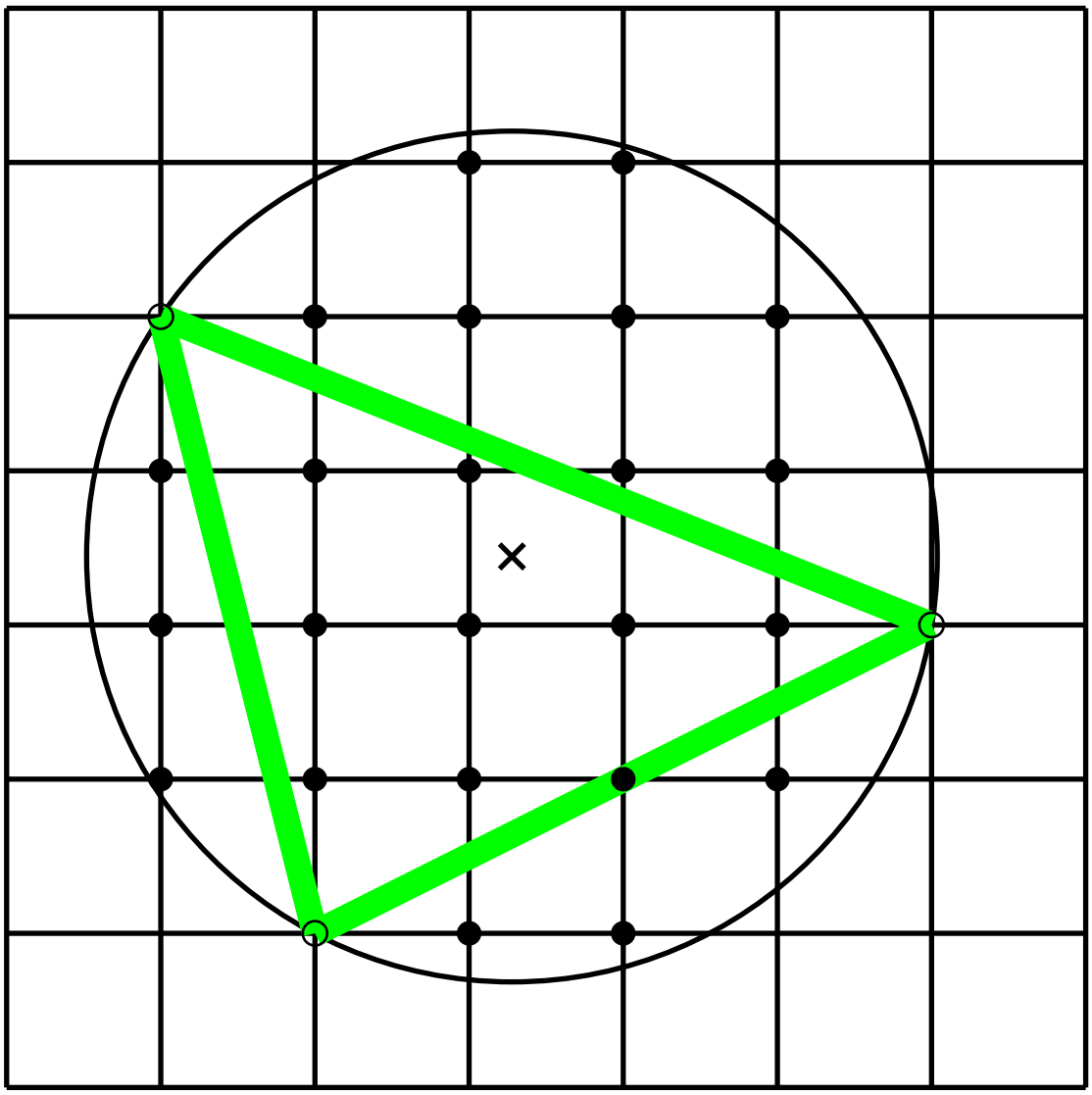
R^2 = 725/98 = 7.39796
R = 2.719919
X = 5/14
Y = 5/14
22 + 4 = 26

A192493(46) = 377, A192494(46) = 50
Triangles: O A



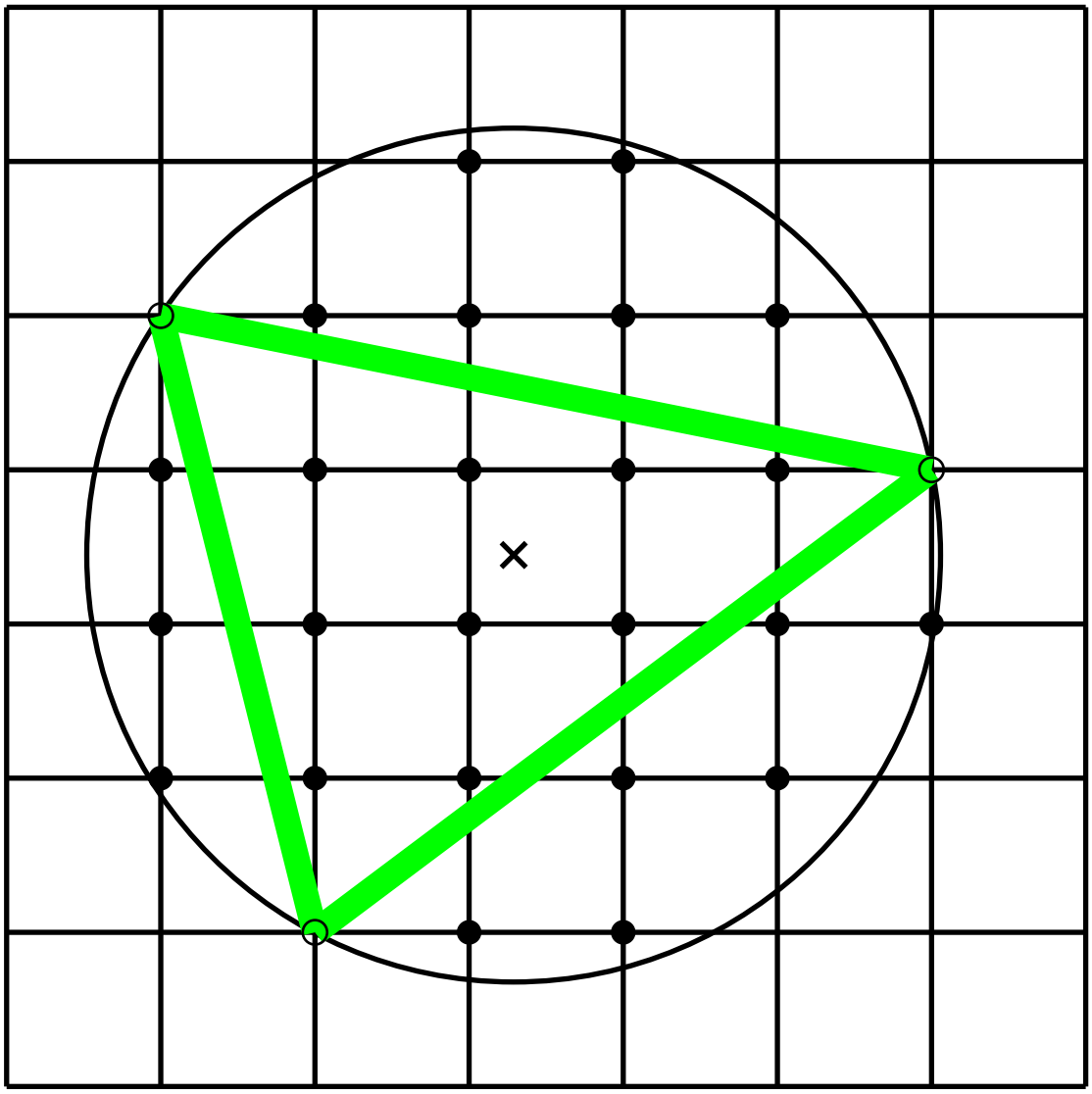
R^2 = 377/ 50 = 7.54000
R = 2.745906
X = 3 / 10
Y = 1 / 2
22 + 4 = 26

A192493(47) = 2465, A192494(47) = 324
Triangles: A



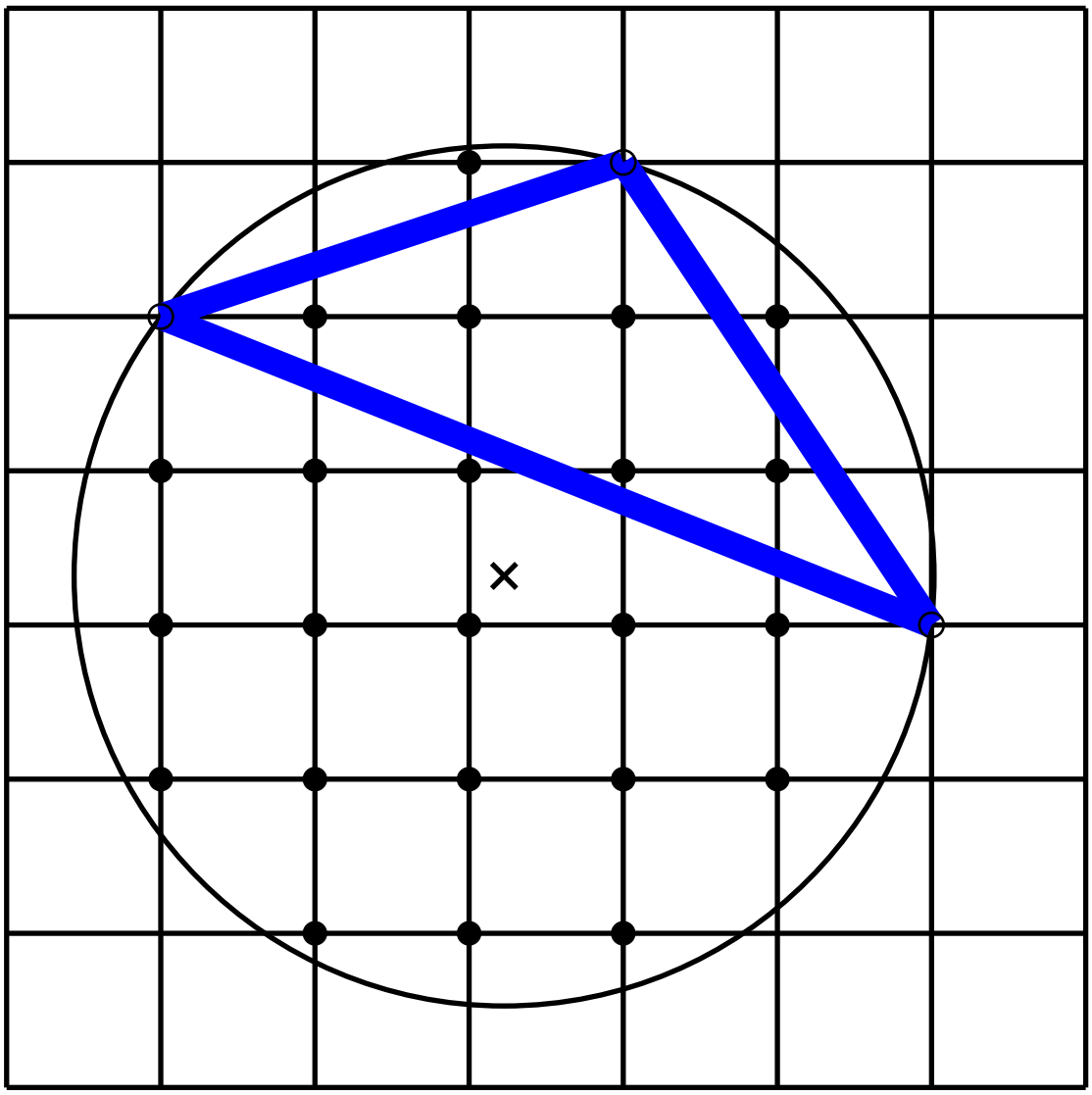
R^2 = 2465/324 = 7.60802
R = 2.758265
X = 5/18
Y = 4/9
23 + 3 = 26

A192493(48) = 5525, A192494(48) = 722
Triangles: A



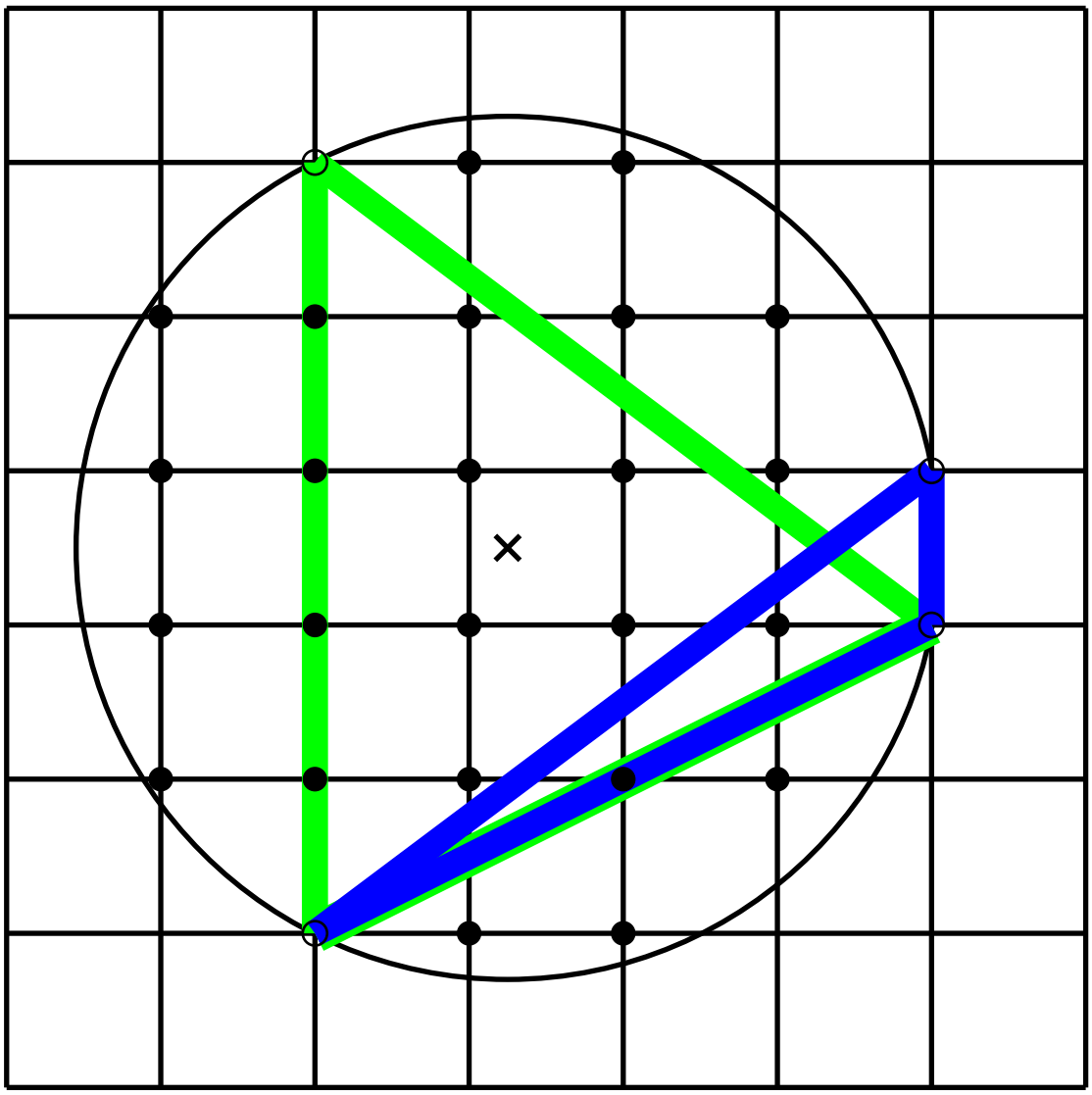
R^2 = 5525/722 = 7.65235
R = 2.766289
X = 11/38
Y = 17/38
24 + 3 = 27

A192493(49) = 1885, A192494(49) = 242
Triangles: O



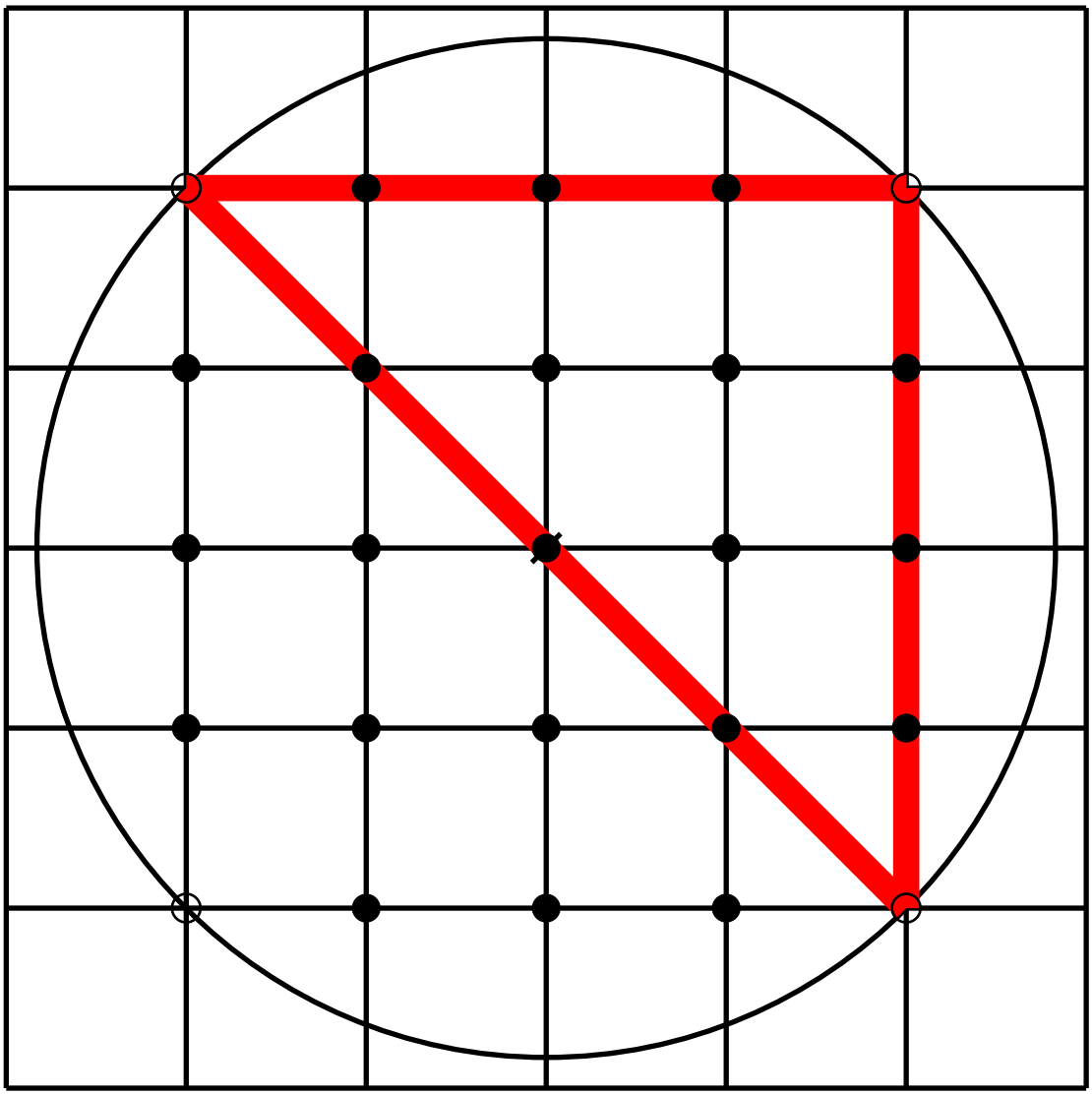
R^2 = 1885/242 = 7.78926
R = 2.790924
X = 5/22
Y = 7/22
23 + 3 = 26

A192493(50) = 125, A192494(50) = 16
Triangles: O A



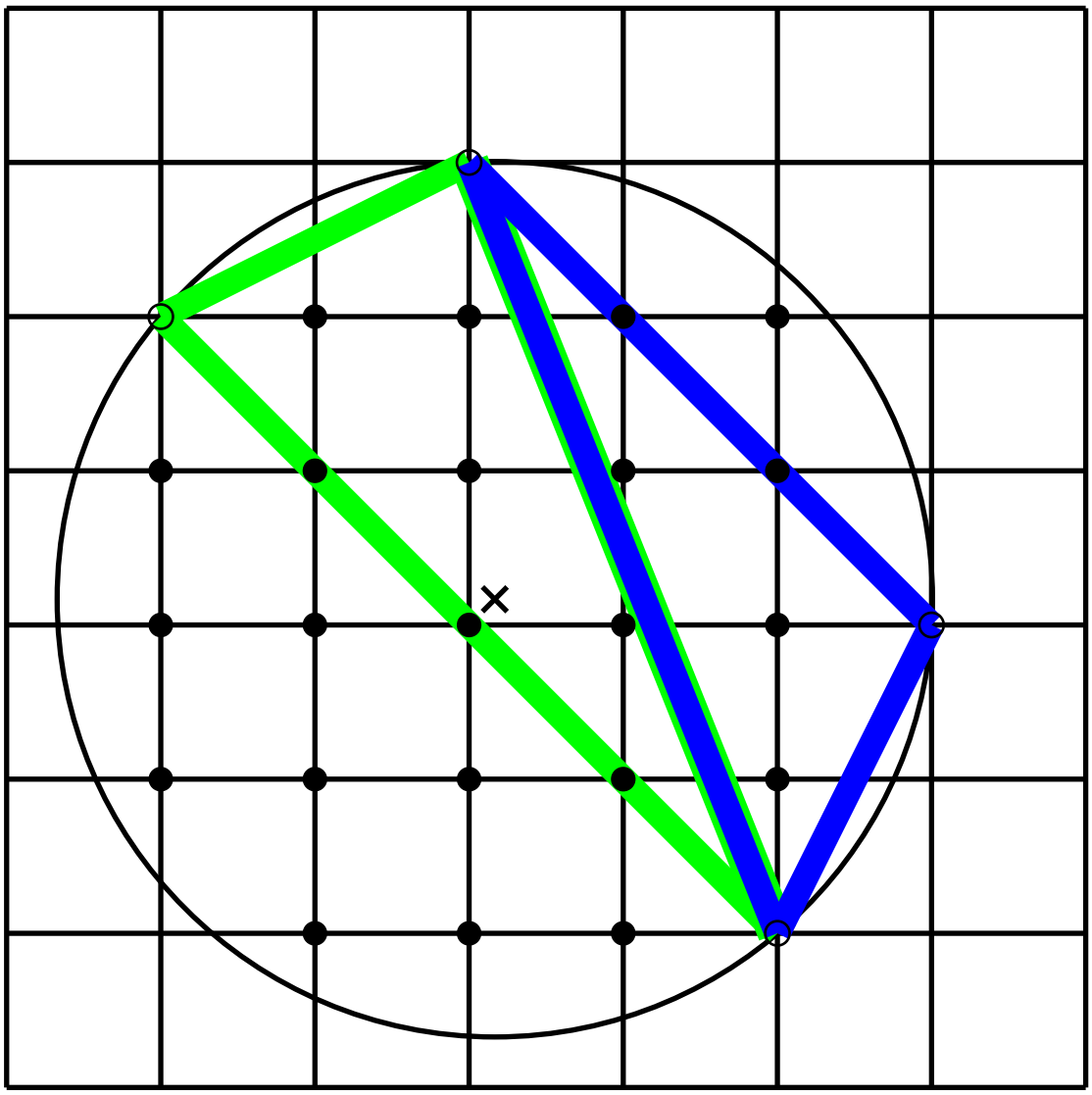
R^2 = 125/16 = 7.81250
R = 2.795085
X = 1/4
Y = 1/2
24 + 4 = 28

A192493(51) = 8, A192494(1) = 1
Triangles: R



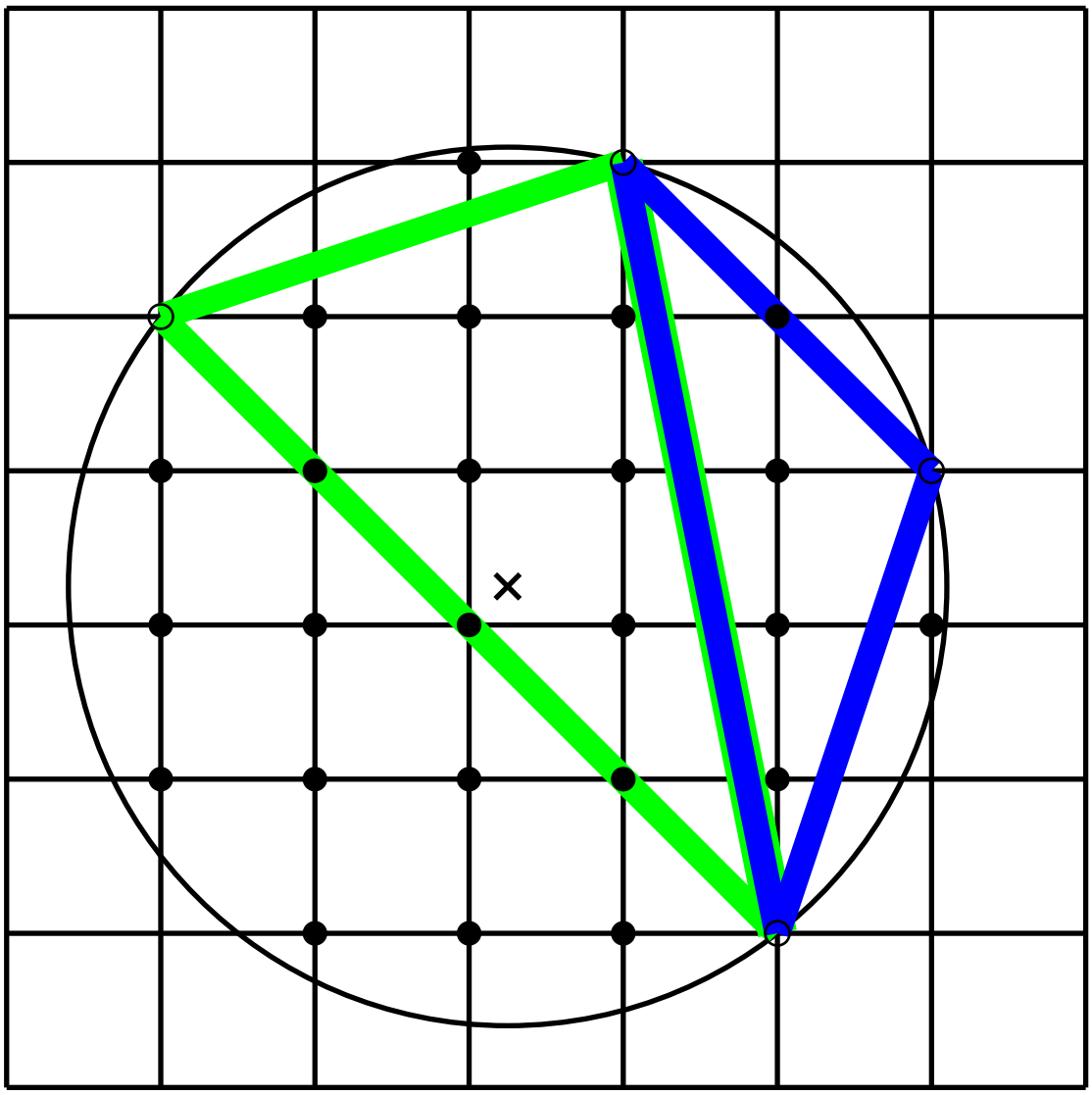
R^2 = 8/1 = 8.00000
R = 2.828427
X = 0/1
Y = 0/1
21 + 4 = 25

A192493(52) = 145, A192494(52) = 18
Triangles: O A



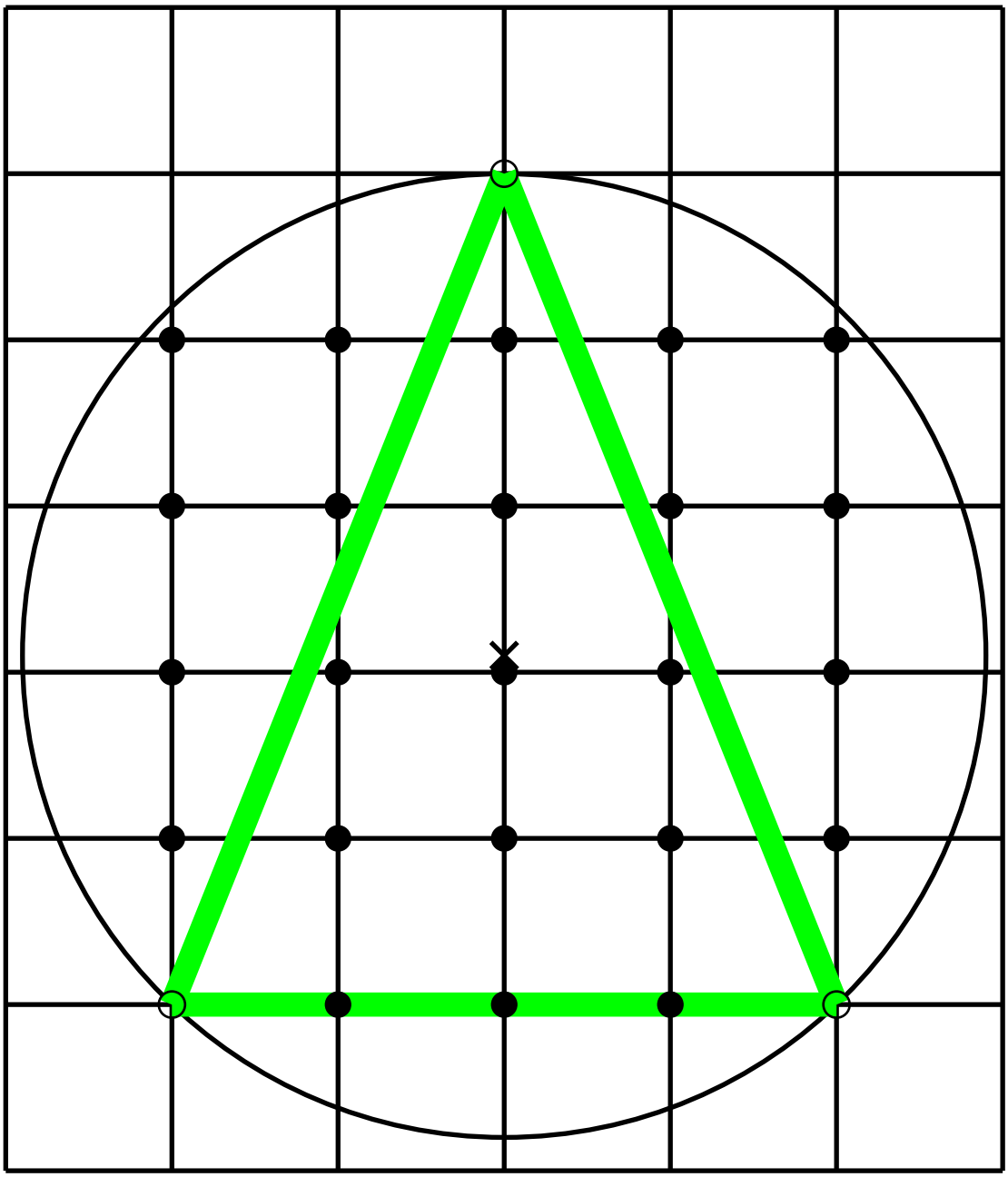
R^2 = 145/18 = 8.05556
R = 2.838231
X = 1/6
Y = 1/6
22 + 4 = 26

A192493(53) = 65, A192494(53) = 8
Triangles: O A



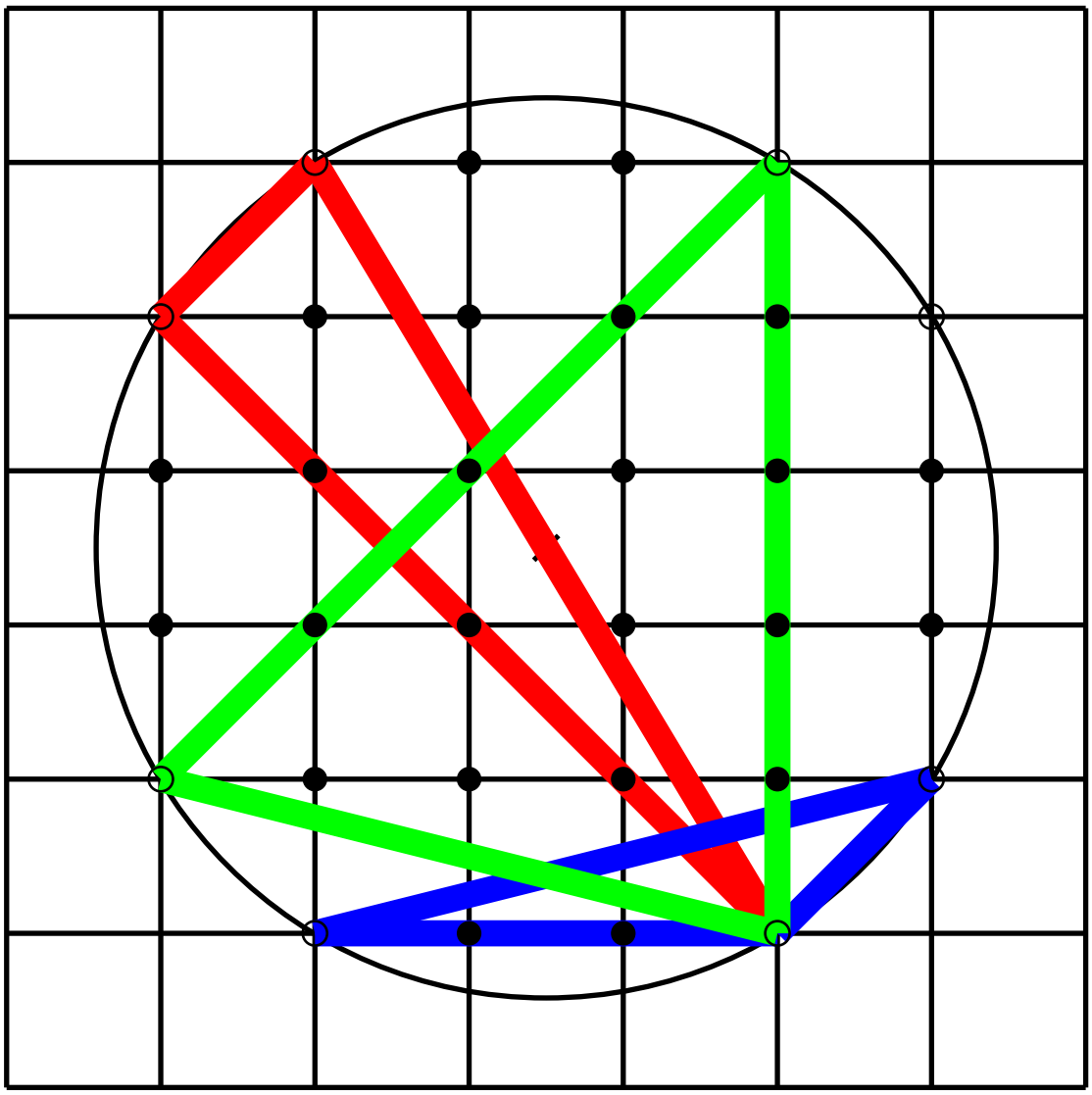
R^2 = 65/8 = 8.12500
R = 2.850439
X = 1/4
Y = 1/4
24 + 4 = 28

A192493(54) = 841, A192494(54) = 100
Triangles: A



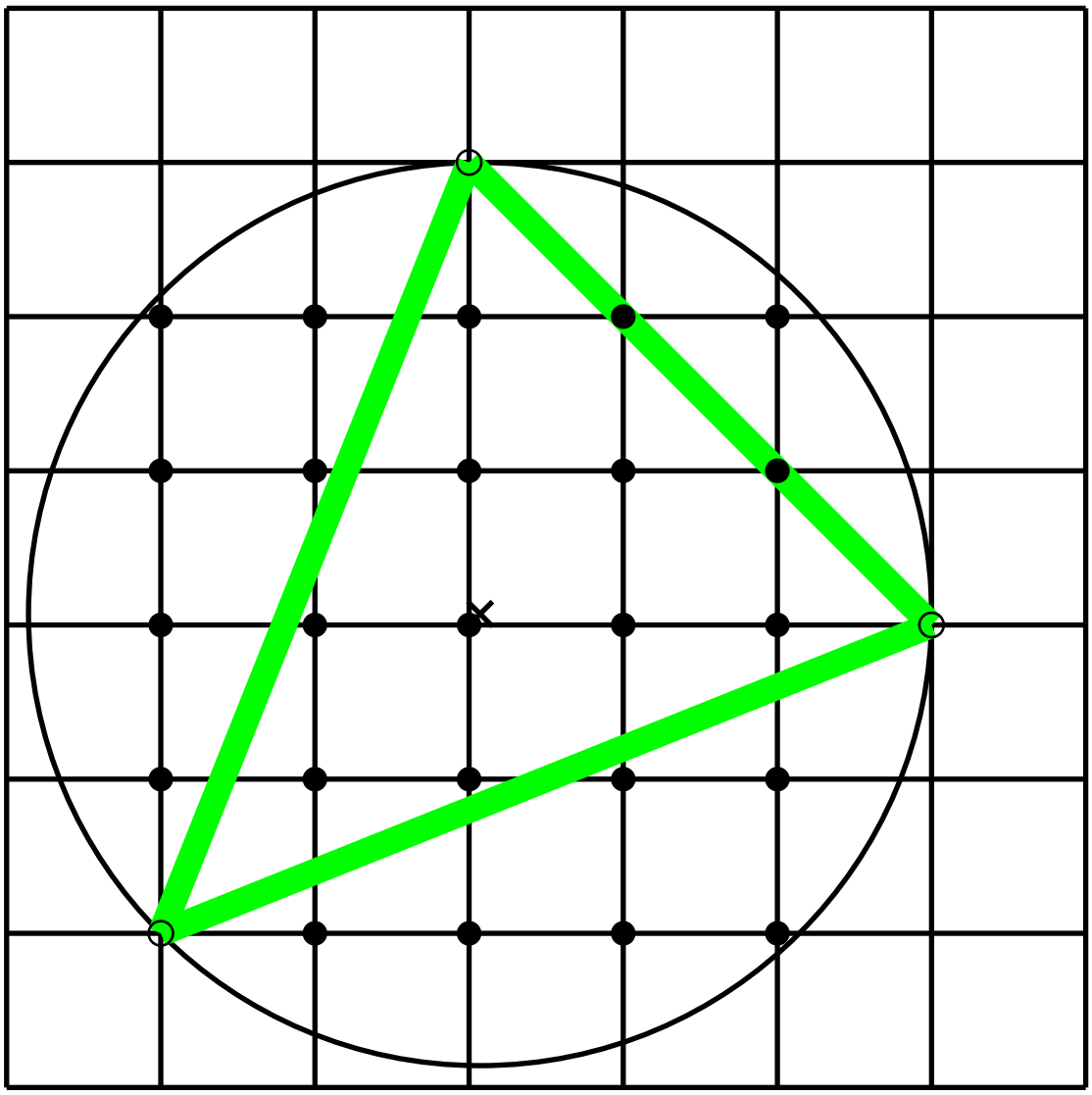
R^2 = 841/100 = 8.41000
R = 2.900000
X = 0/1
Y = 1/10
23 + 3 = 26

A192493(55) = 17, A192494(55) = 2
Triangles: O R A



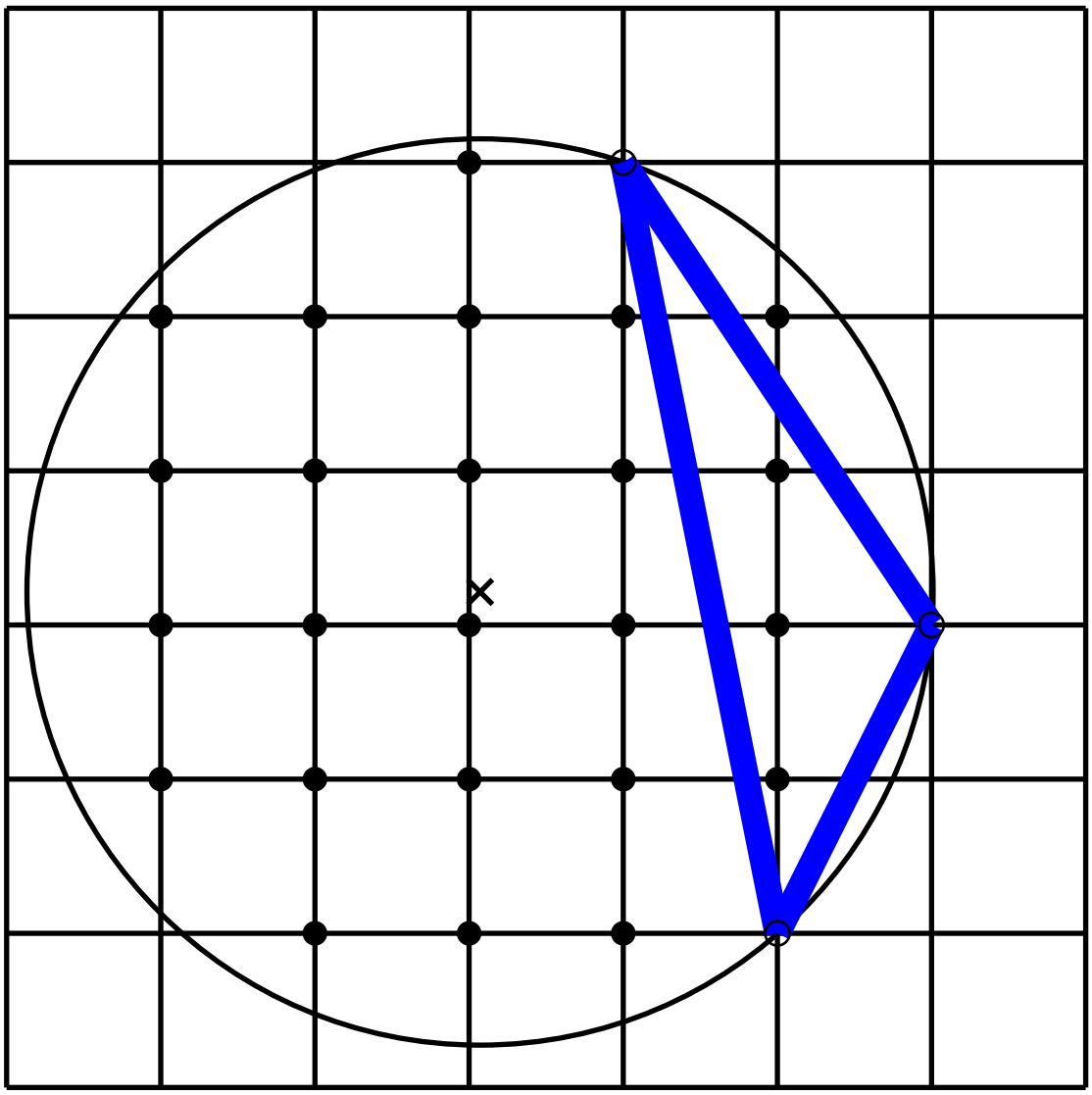
R^2 = 17/2 = 8.50000
R = 2.915476
X = 1/2
Y = 1/2
24 + 8 = 32

A192493(56) = 841, A192494(14) = 98
Triangles: A



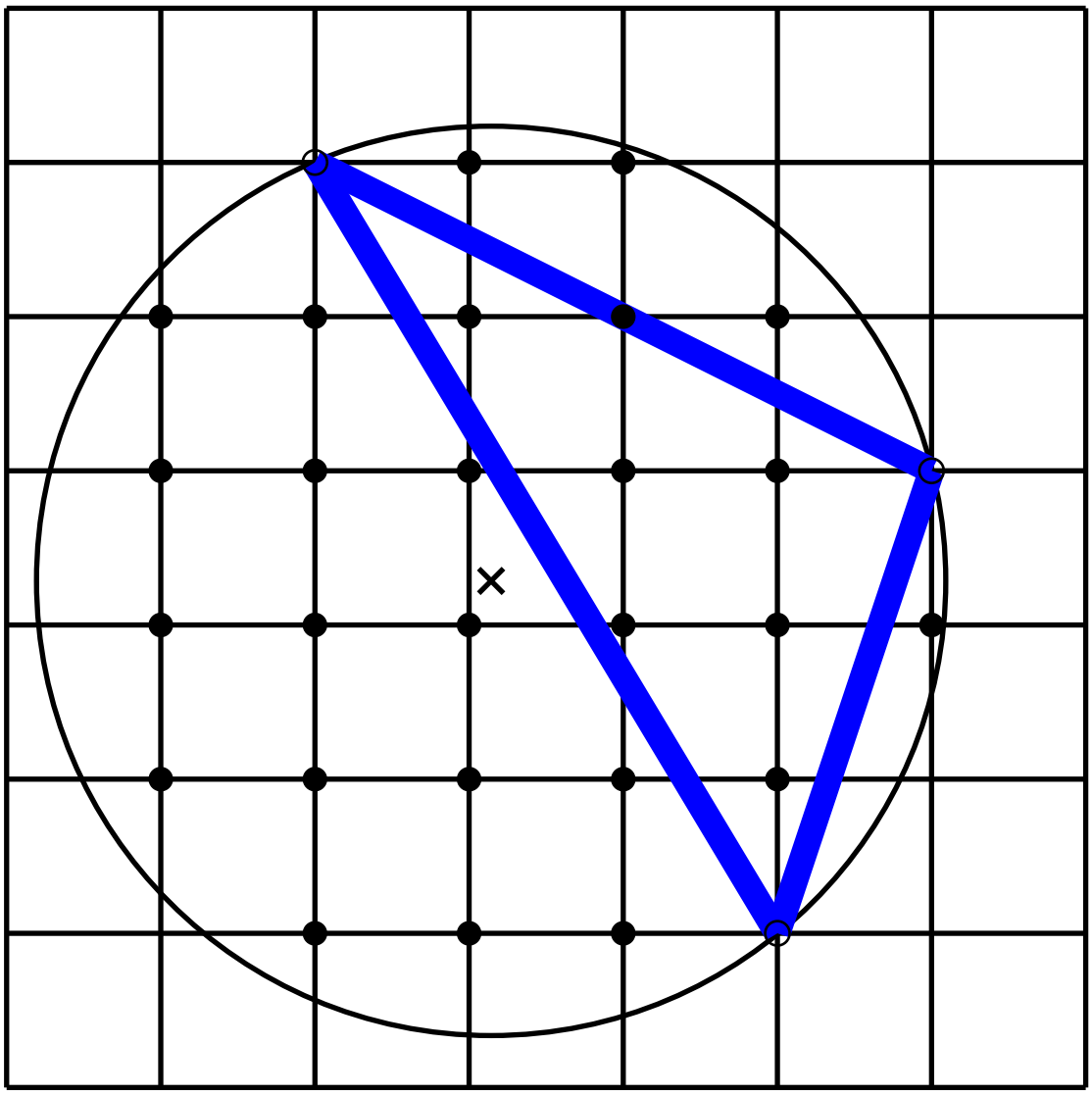
R^2 = 841/98 = 8.58163
R = 2.929442
X = 1/14
Y = 1/14
24 + 3 = 27

A192493(57) = 845, A192494(57) = 98
Triangles: O



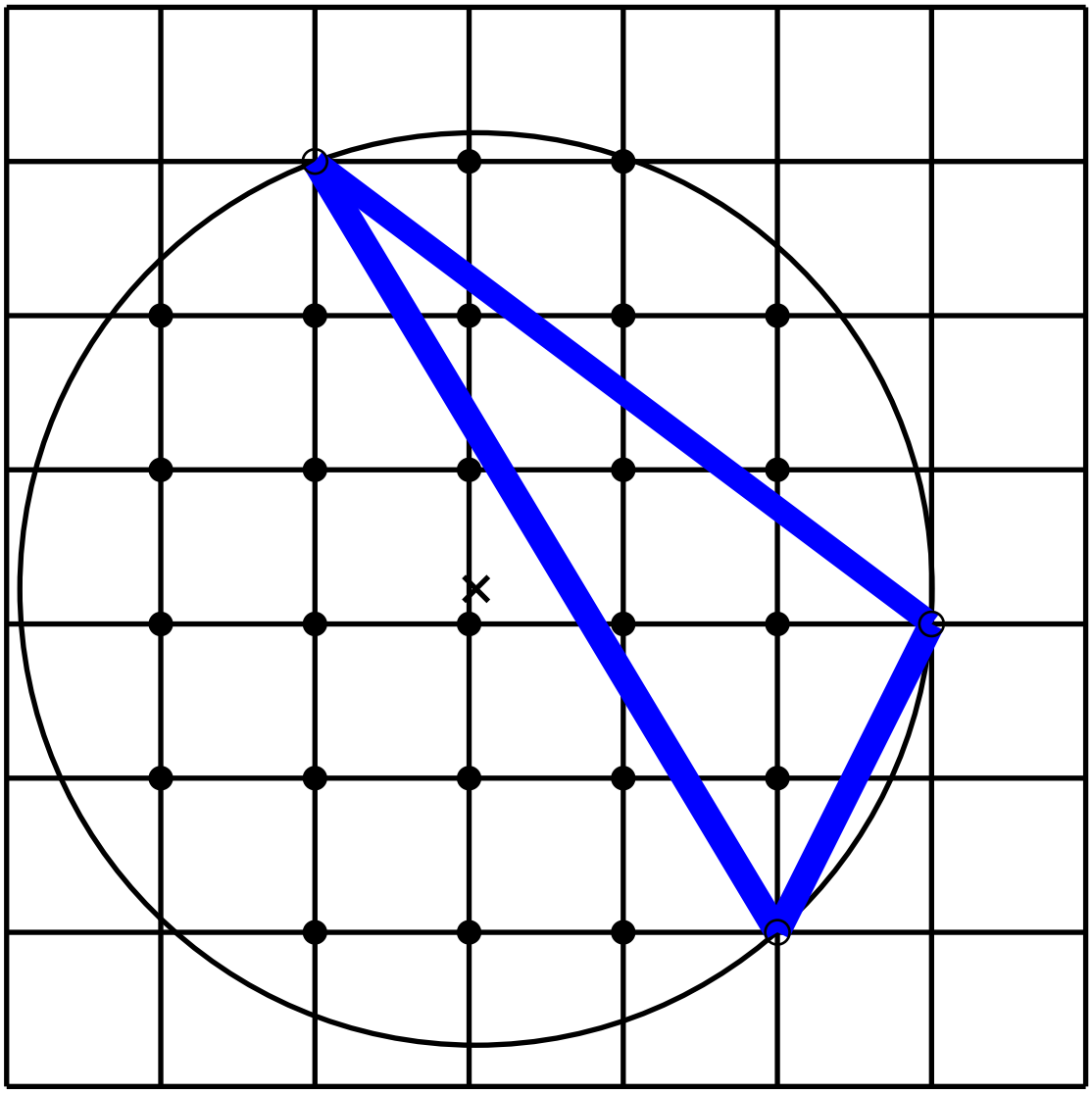
R^2 = 845/98 = 8.62245
R = 2.936401
X = 1/14
Y = 3/14
24 + 3 = 27

A192493(58) = 425, A192494(58) = 49
Triangles: O



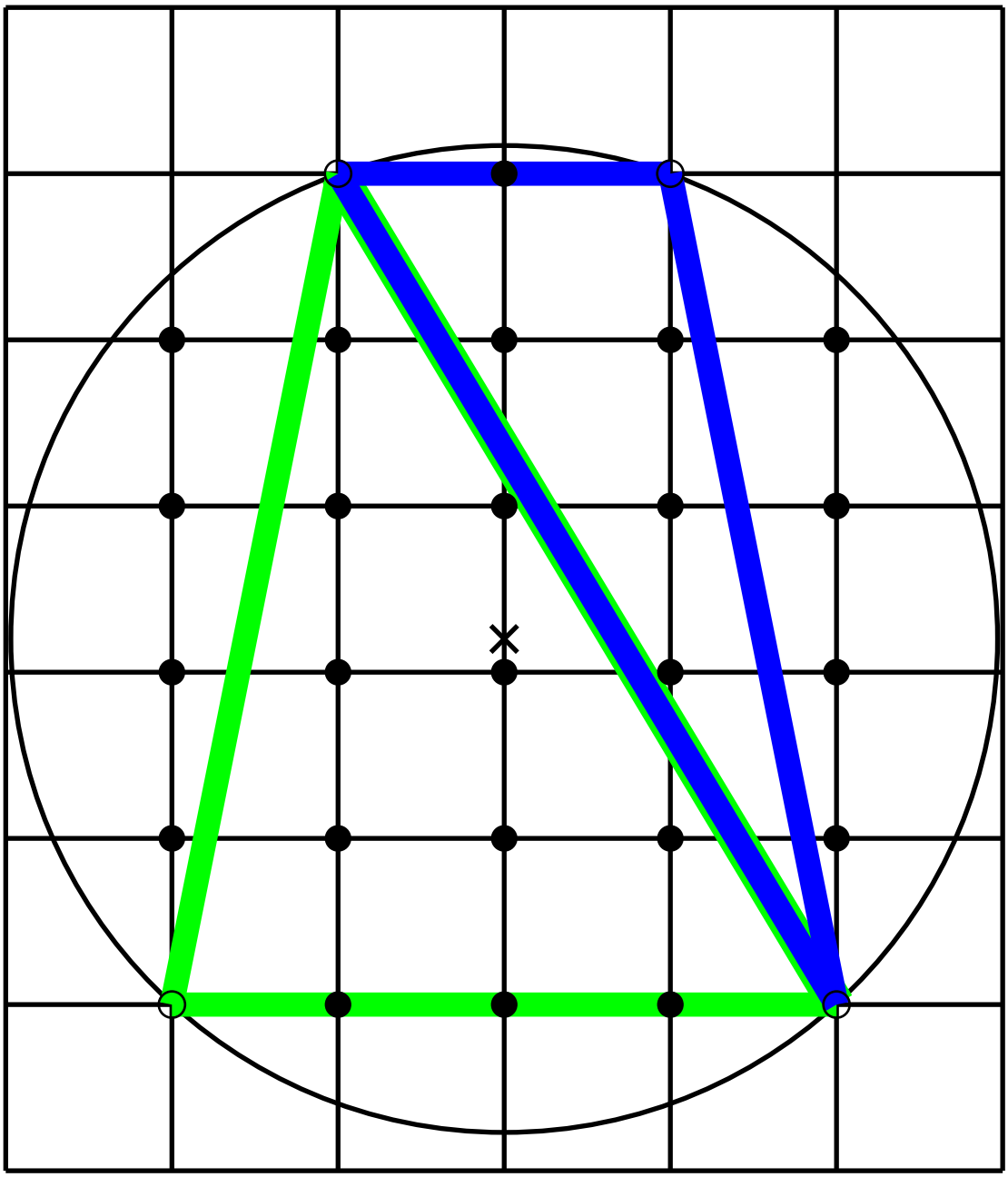
R^2 = 425/49 = 8.67347
R = 2.945075
X = 1/7
Y = 2/7
26 + 3 = 29

A192493(59) = 2125, A192494(59) = 49
Triangles: O



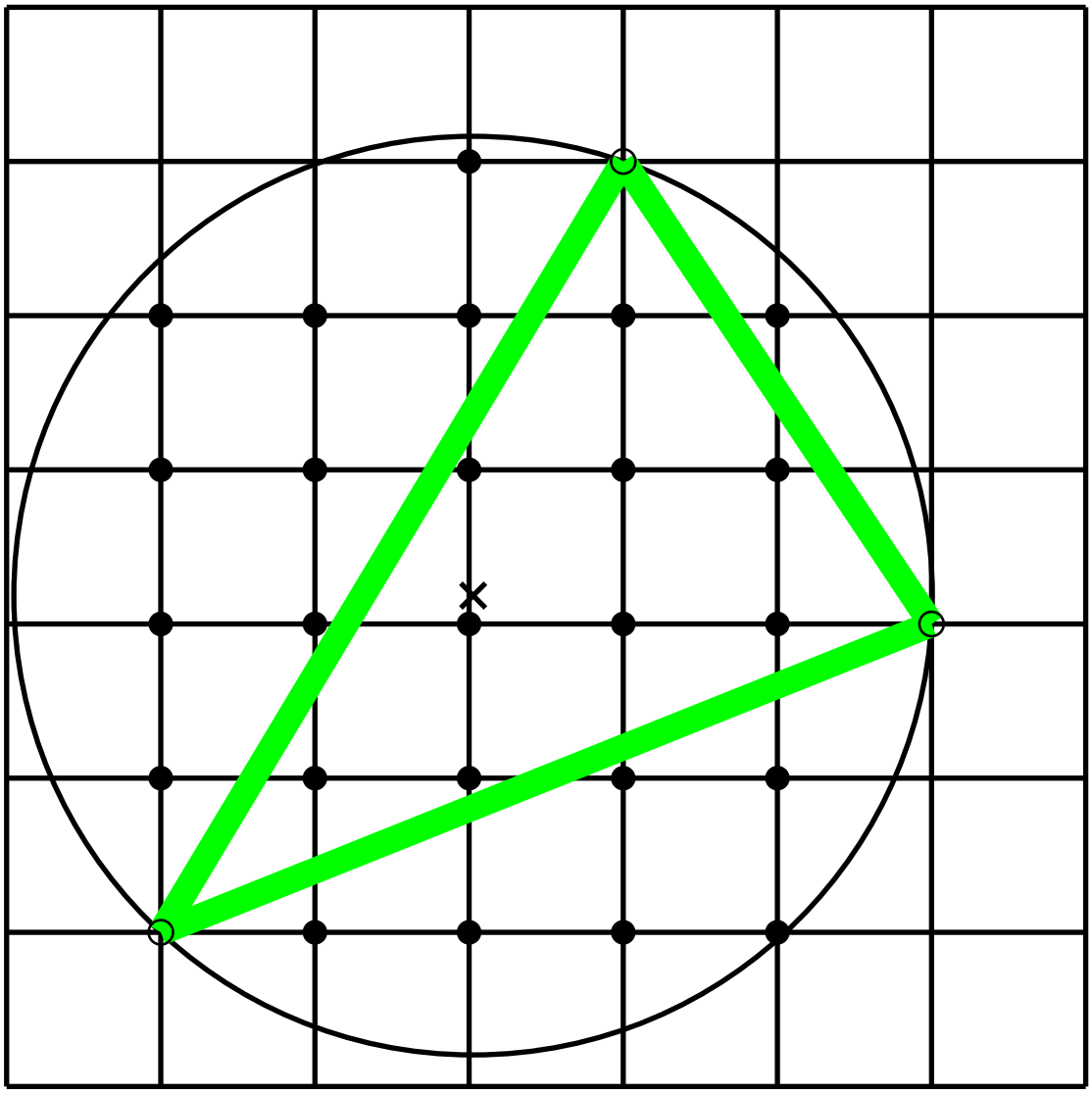
R^2 = 2125/242 = 8.78099
R = 2.963274
X = 1/22
Y = 5/22
25 + 3 = 28

A192493(60) = 221, A192494(60) = 25
Triangles: O A



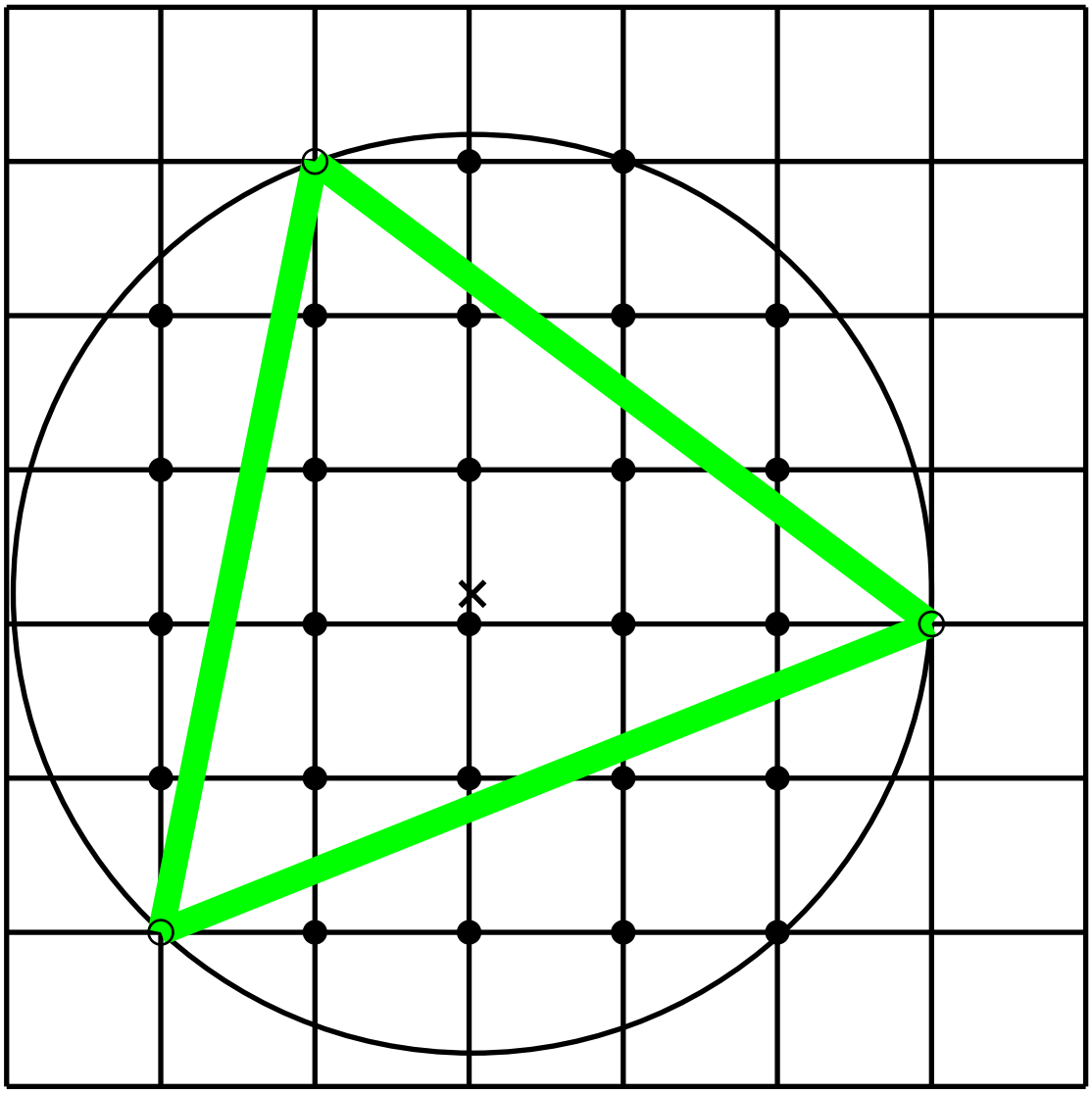
R^2 = 221/25 = 8.84000
R = 2.973214
X = 0/1
Y = 1/5
24 + 4 = 28

A192493(61) = 6409, A192494(61) = 722
Triangles: A



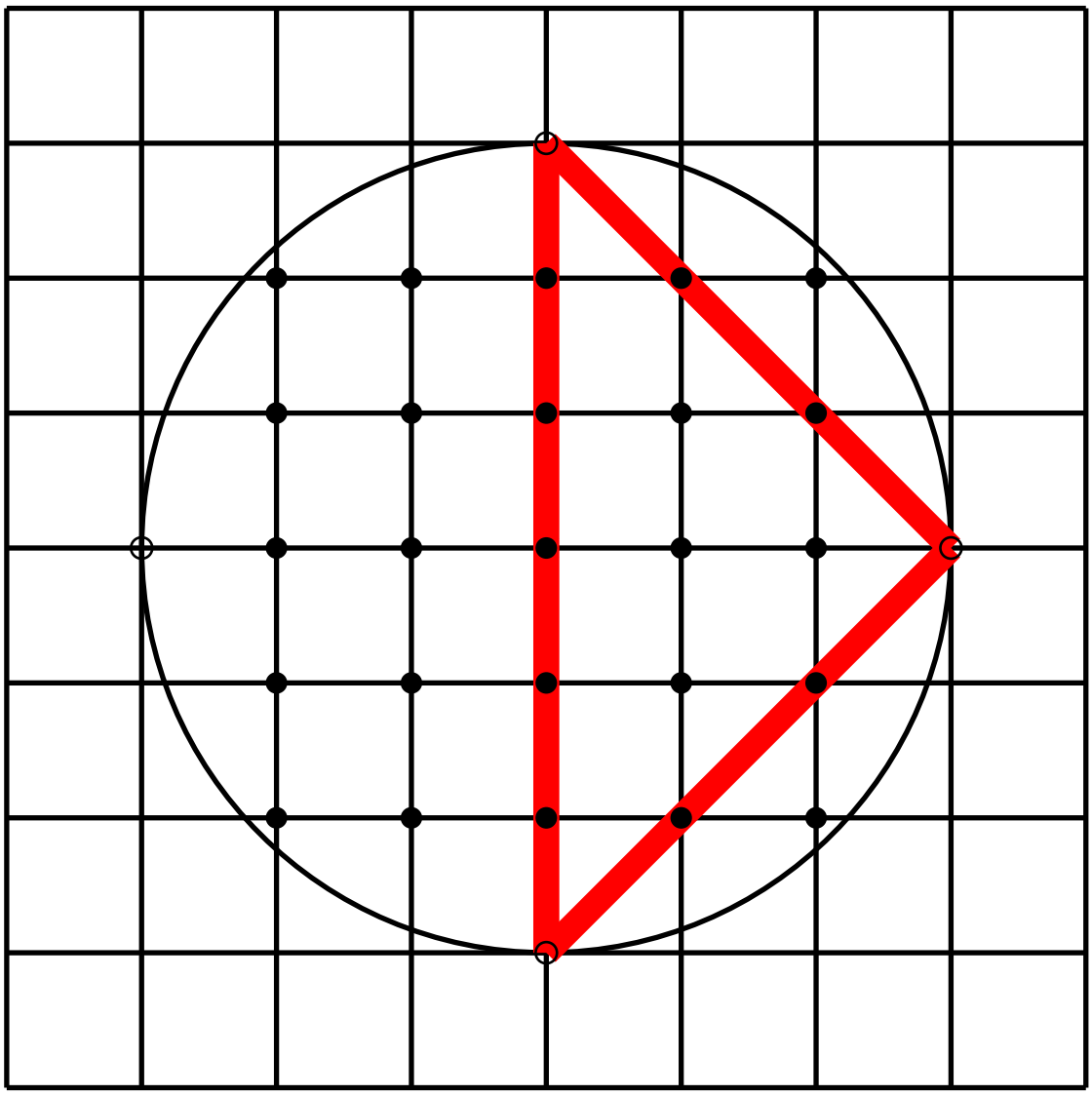
R^2 = 6409/722 = 8.87673
R = 2.979384
X = 1/38
Y = 7/38
25 + 3 = 28

A192493(62) = 9425, A192494(62) = 1058
Triangles: A



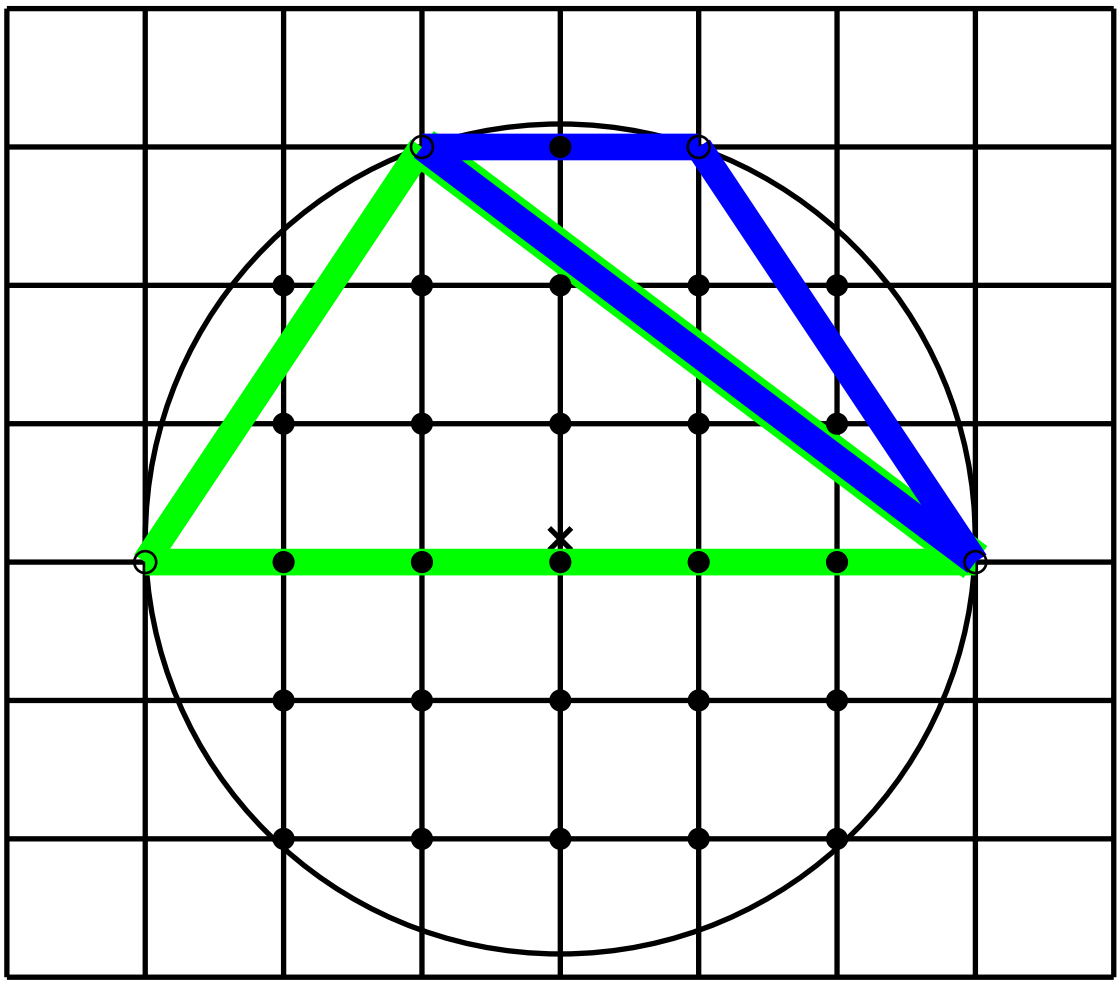
R^2 = 9425/1058 = 8.90832
R = 2.984680
X = 1/46
Y = 9/46
26 + 3 = 29

A192493(63) = 9, A192494(63) = 1
Triangles: R



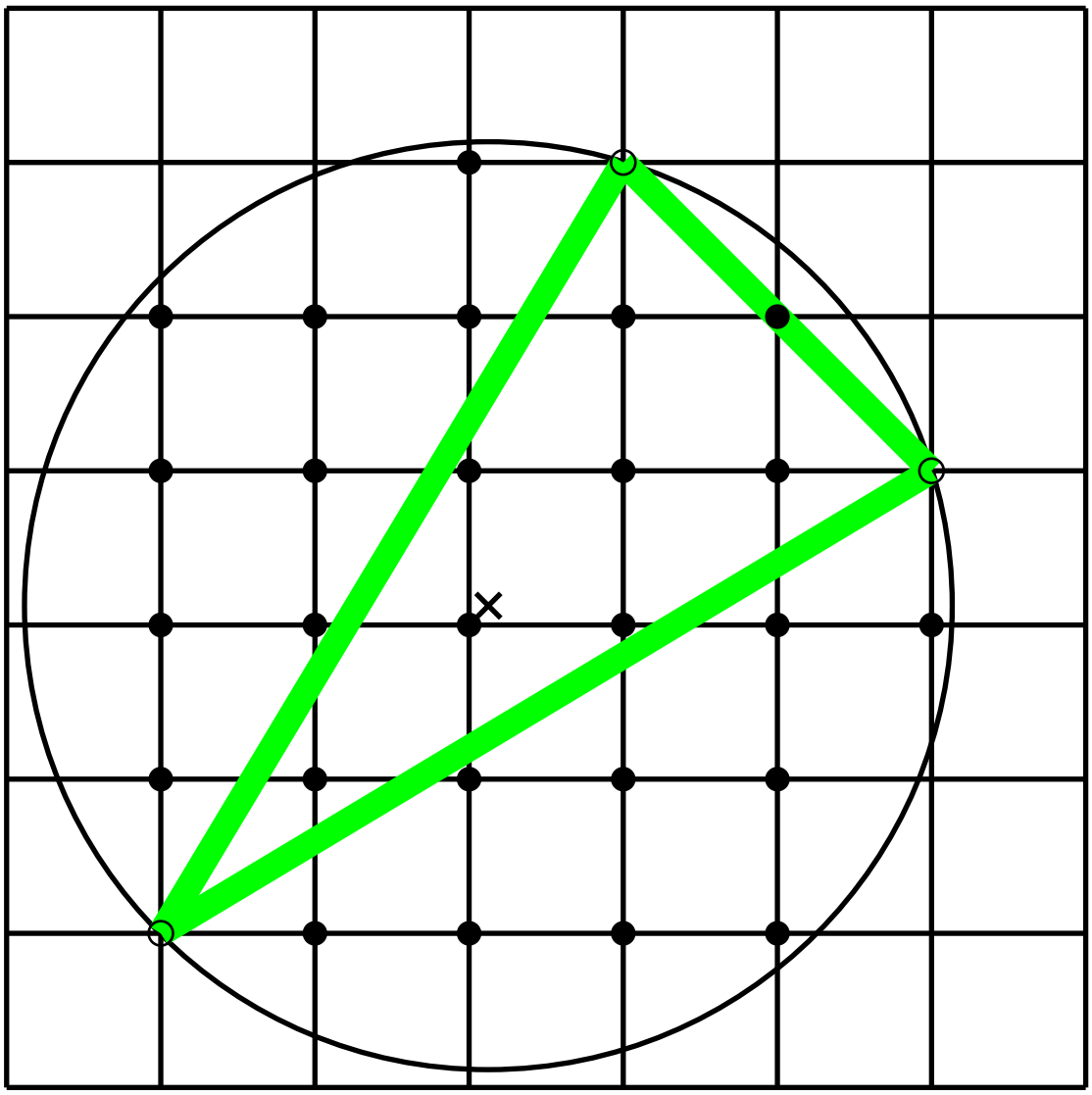
R^2 = 9/1 = 9.00000
R = 3.000000
X = 0/1
Y = 0/1
25 + 4 = 29

A192493(64) = 325, A192494(64) = 36
Triangles: O A



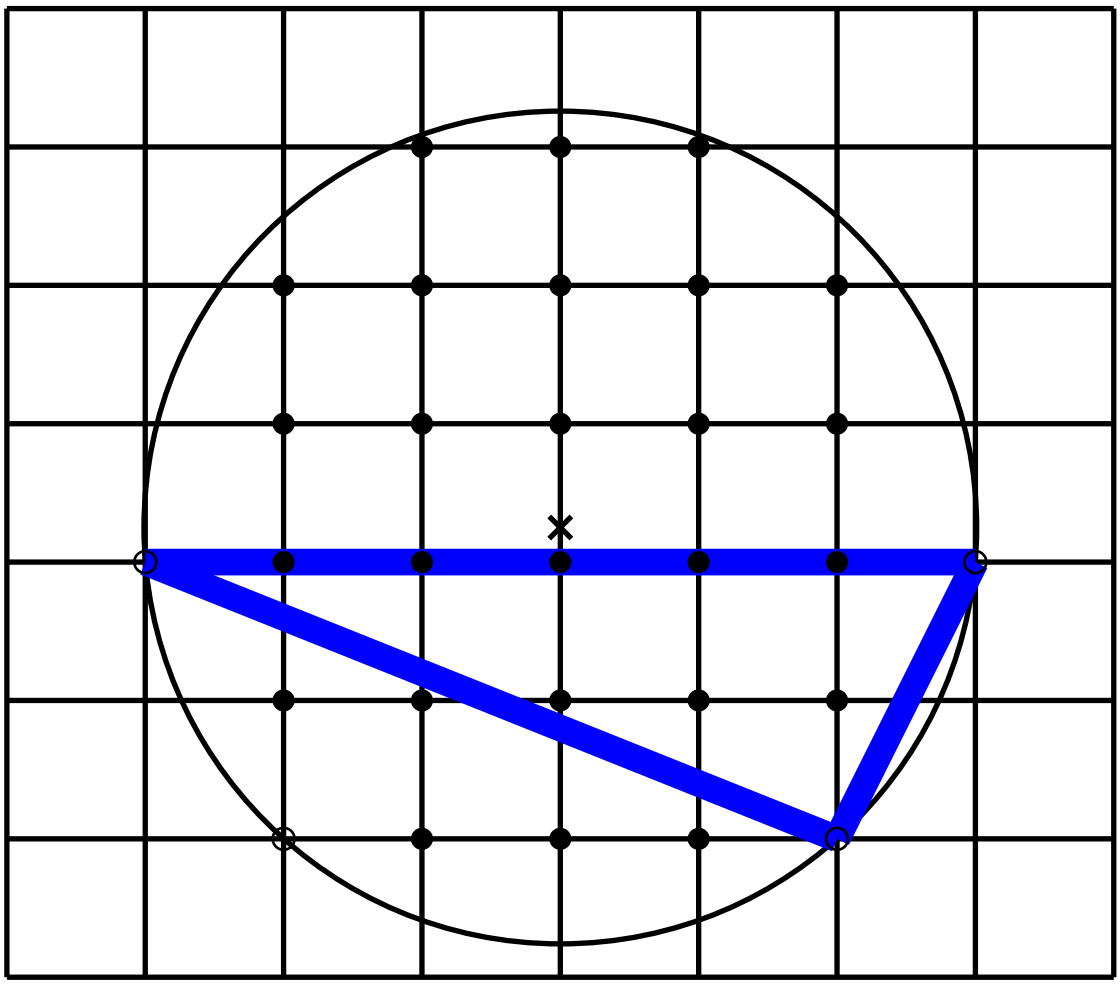
R^2 = 325/36 = 9.02778
R = 3.004626
X = 0/1
Y = 1/6
26 + 4 = 30

A192493(65) = 289, A192494(65) = 32
Triangles: A



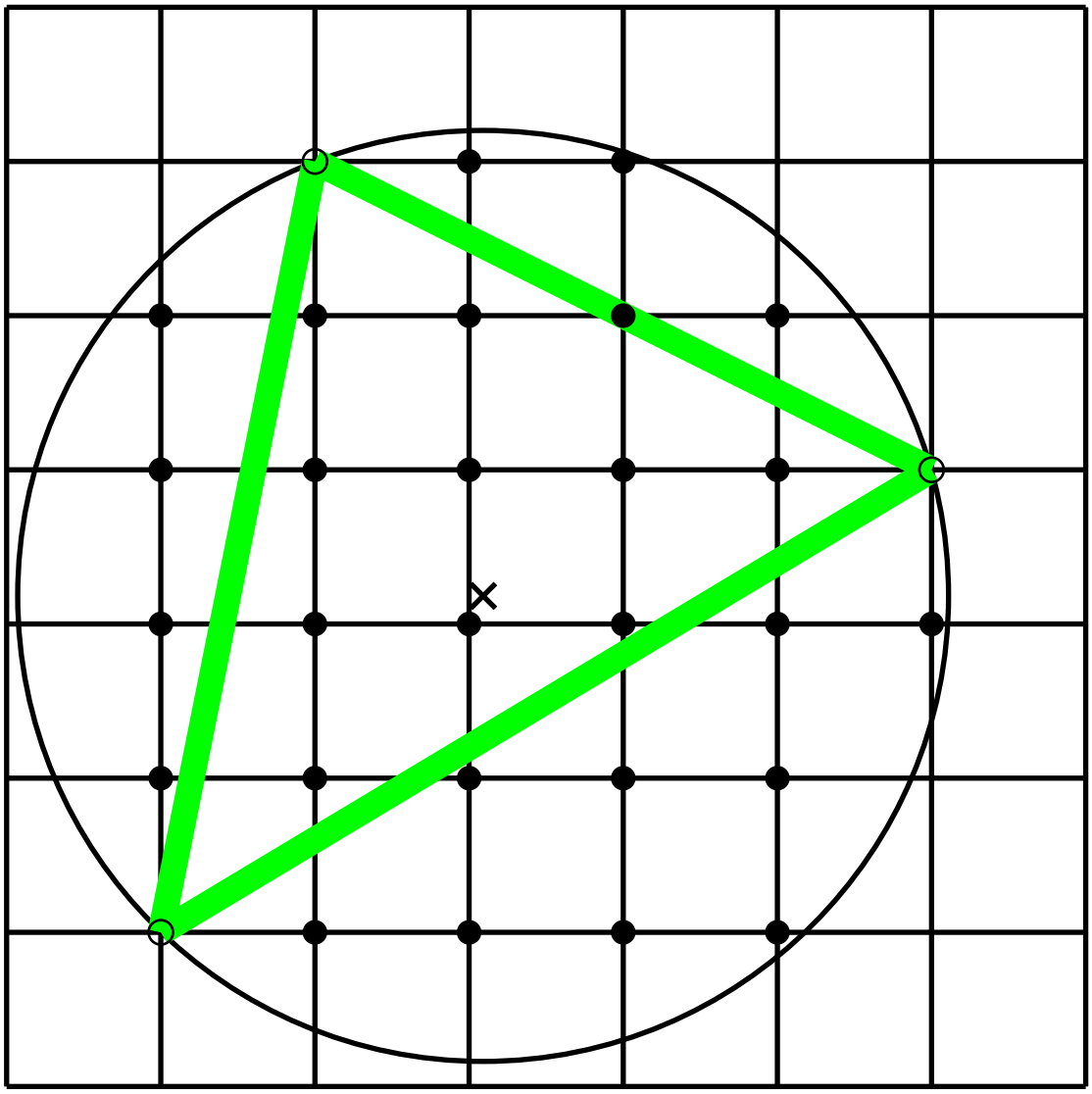
R^2 = 289/32 = 9.03125
R = 3.005204
X = 1/8
Y = 1/8
26 + 3 = 29

A192493(66) = 145, A192494(66) = 16
Triangles: O



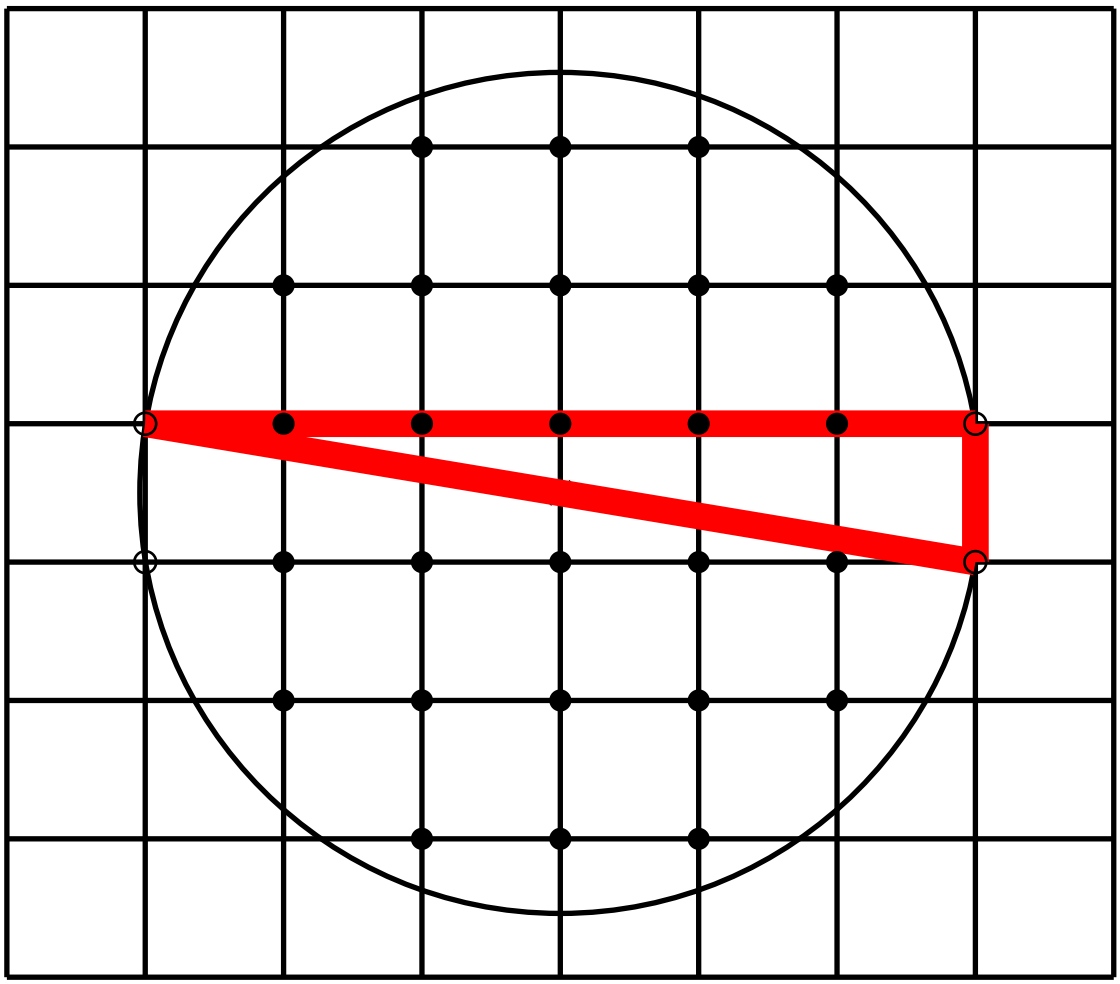
R^2 = 145/16 = 9.06250
R = 3.010399
X = 0/1
Y = 1/4
26 + 4 = 30

A192493(67) = 1105, A192494(67) = 121
Triangles: A



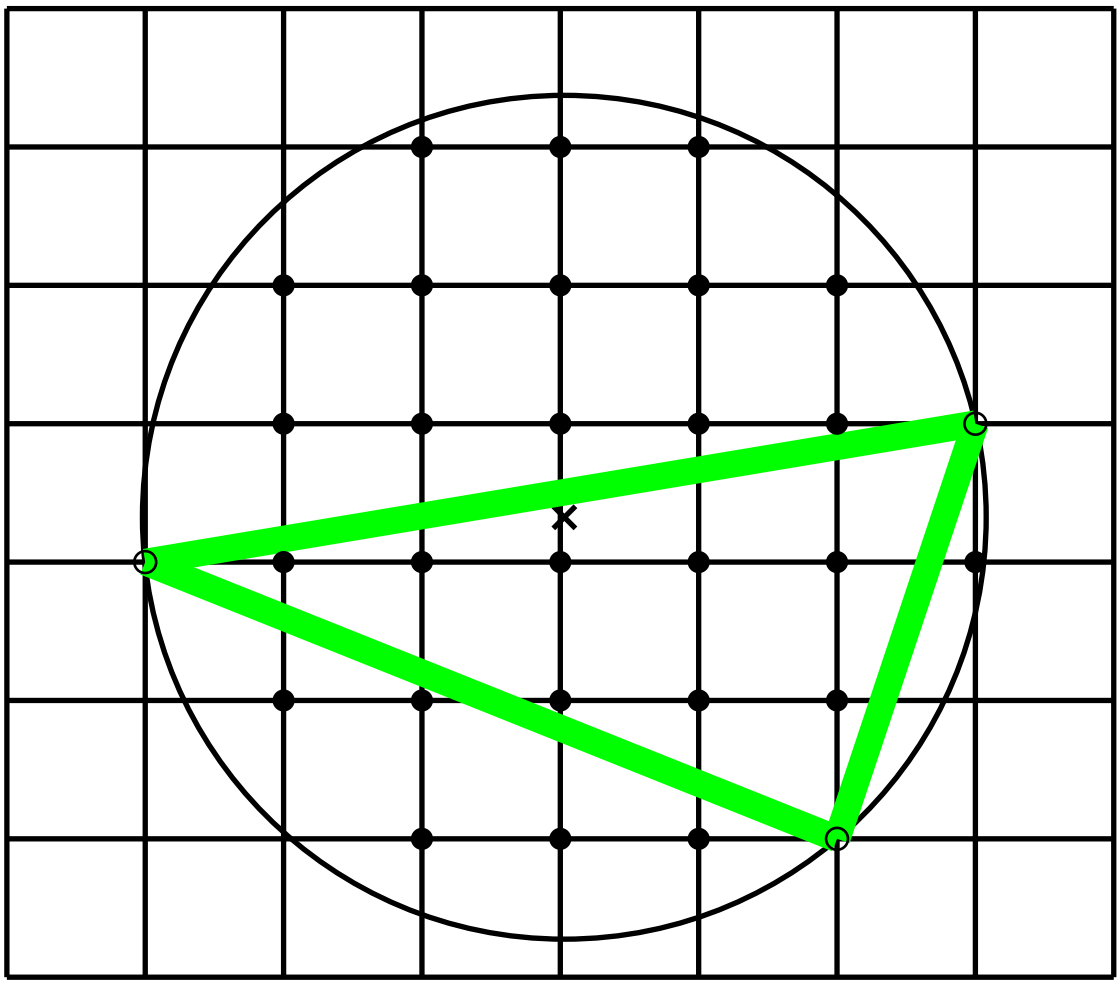
R^2 = 1105/121 = 9.13223
R = 3.021958
X = 1/11
Y = 2/11
27 + 3 = 30

A192493(68) = 37, A192494(68) = 4
Triangles: R



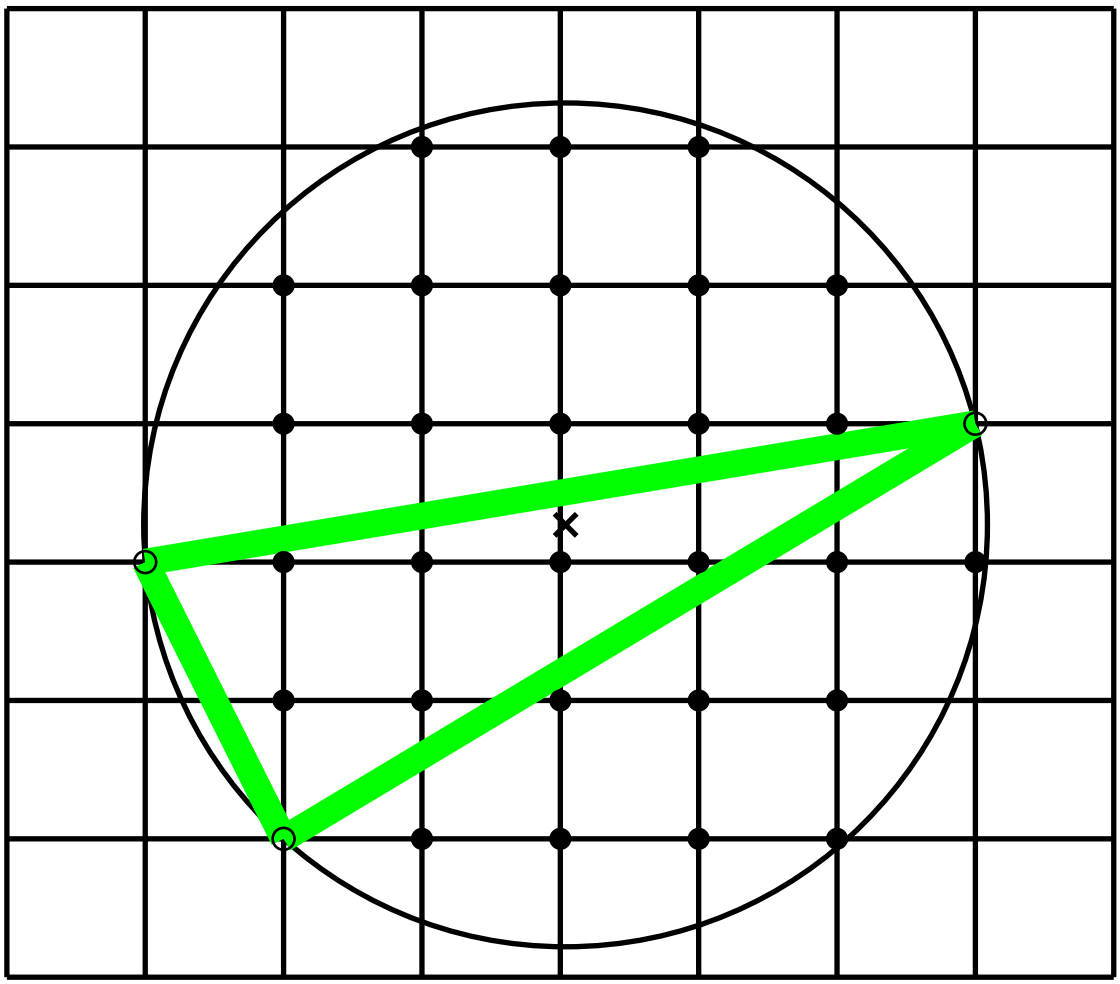
R^2 = 37/4 = 9.25000
R = 3.041381
X = 0/1
Y = 1/2
26 + 4 = 30

A192493(69) = 5365, A192494(69) = 578
Triangles: A



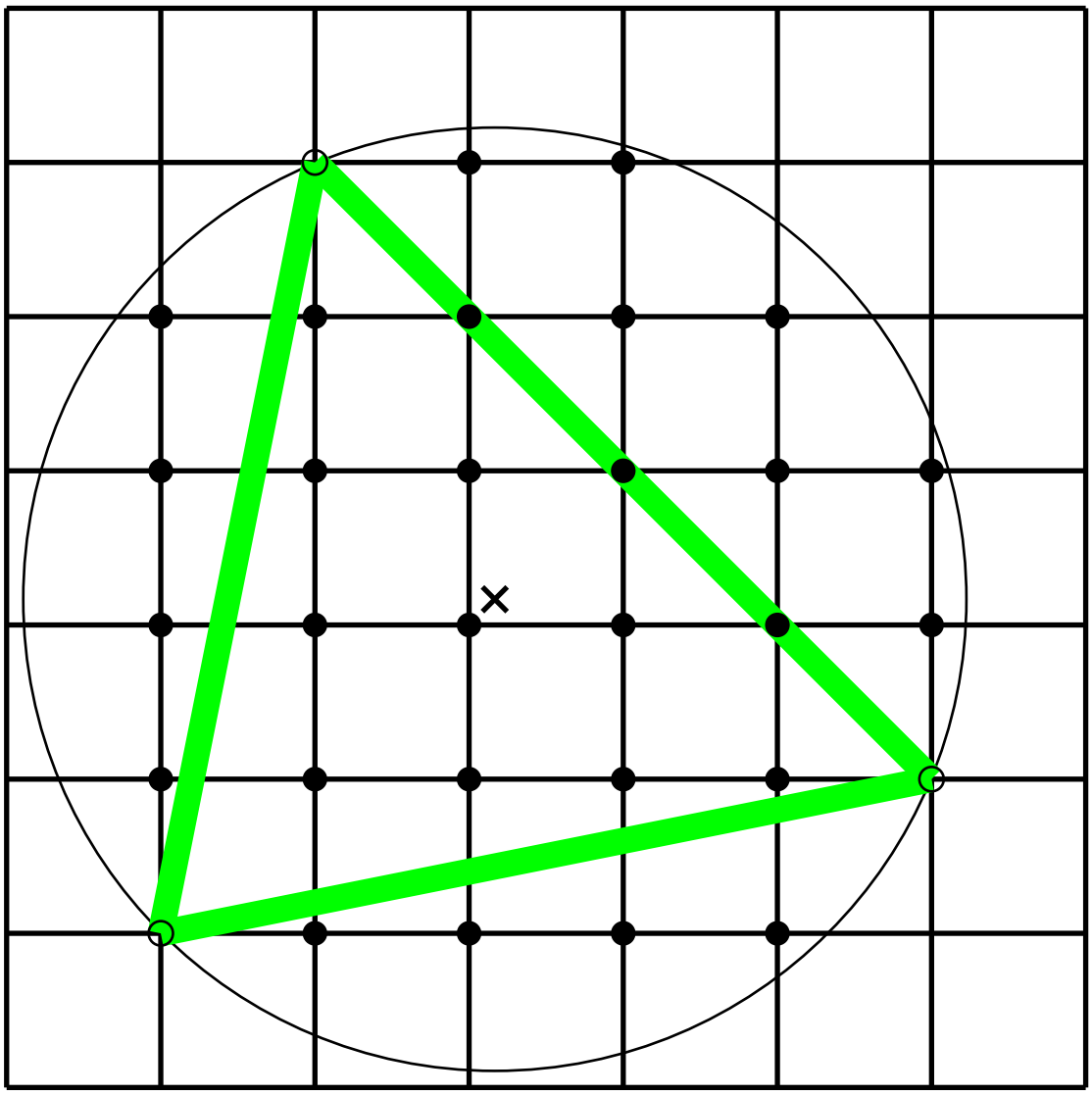
R^2 = 5365/578 = 9.28201
R = 3.046639
X = 1/34
Y = 11/34
27 + 3 = 30

A192493(70) = 3145, A192494(70) = 338
Triangles: A



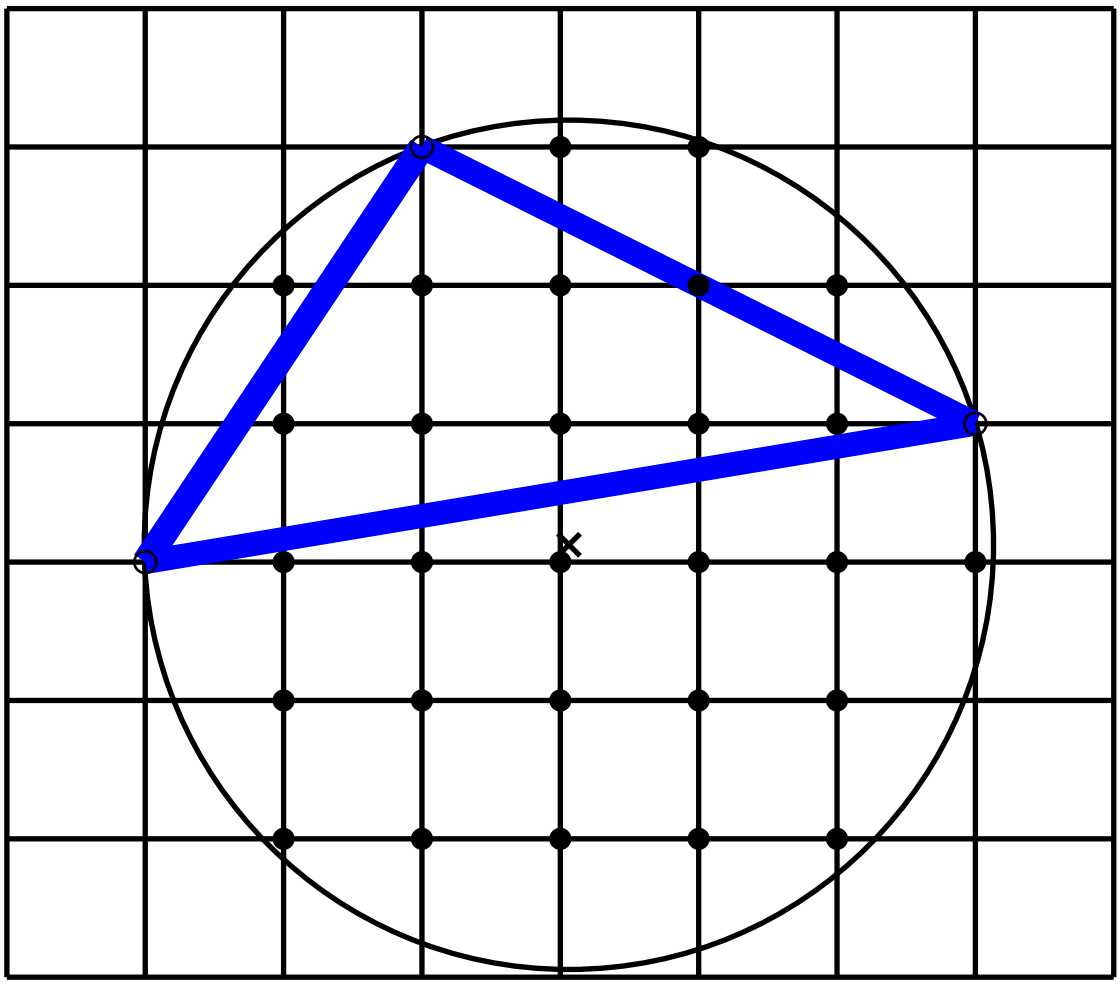
R^2 = 3145/ 338 = 9.30473
R = 3.050366
X = 1/26
Y = 7/26
28 + 3 = 31

A192493(71) = 169, A192494(71) = 18
Triangles: A



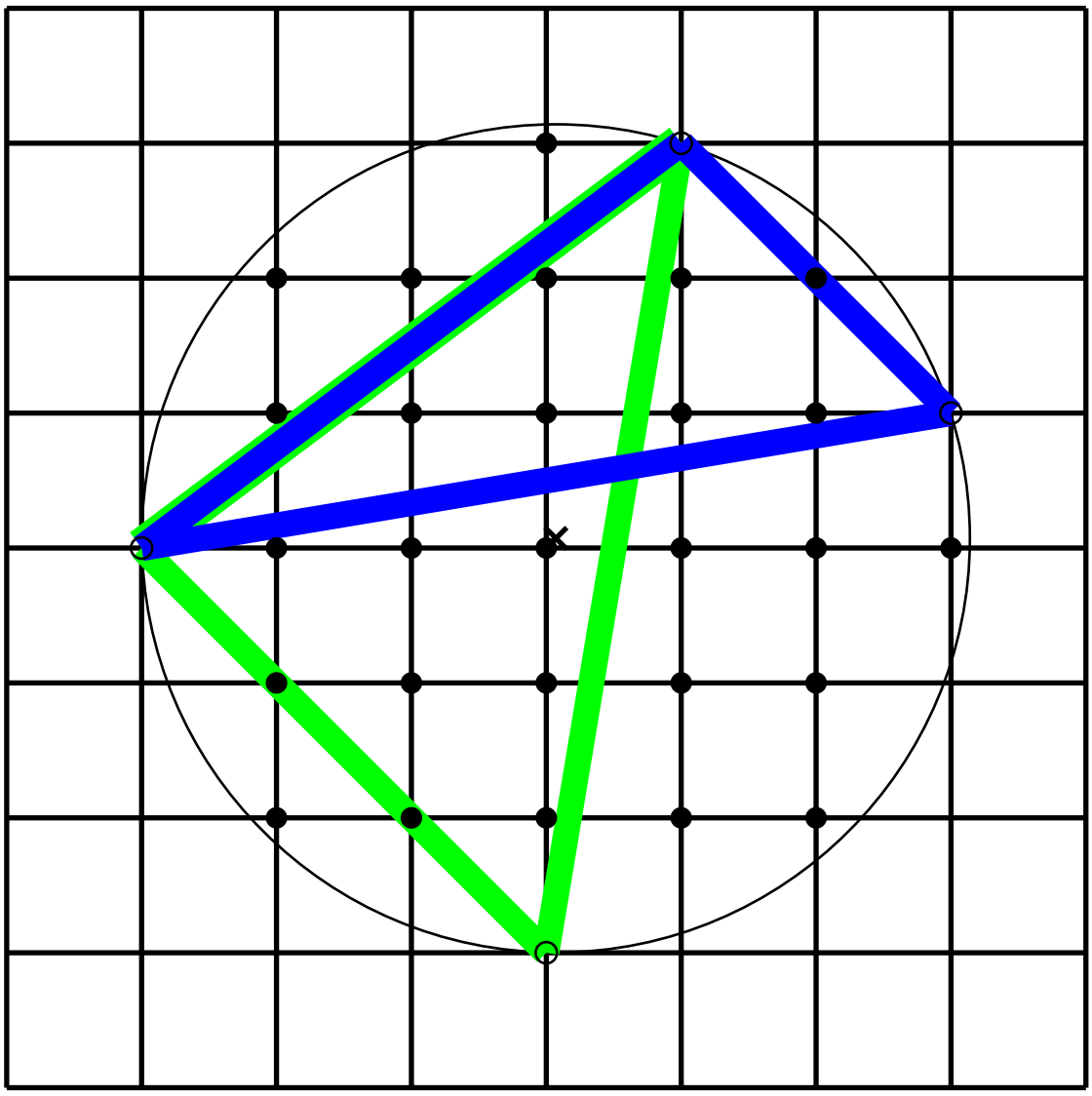
R^2 = 169/ 18 = 9.38889
R = 3.064129
X = 1 / 6
Y = 1 / 6
28 + 3 = 31

A192493(72) = 2405, A192494(72) = 256
Triangles: O



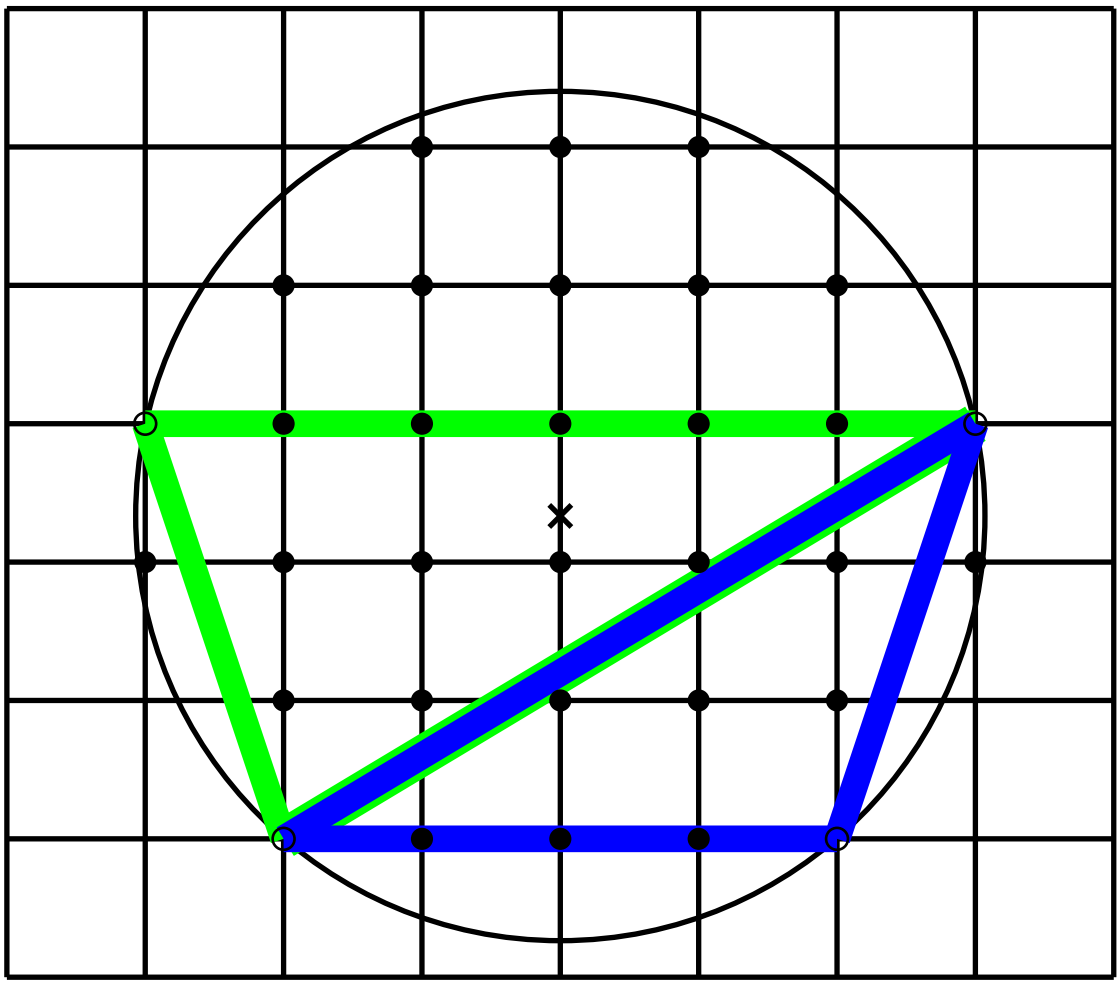
R^2 = 2405/256 = 9.39453
R = 3.065050
X = 1/16
Y = 1/8
28 + 3 = 31

A192493(73) = 925, A192494(73) = 98
Triangles: O A



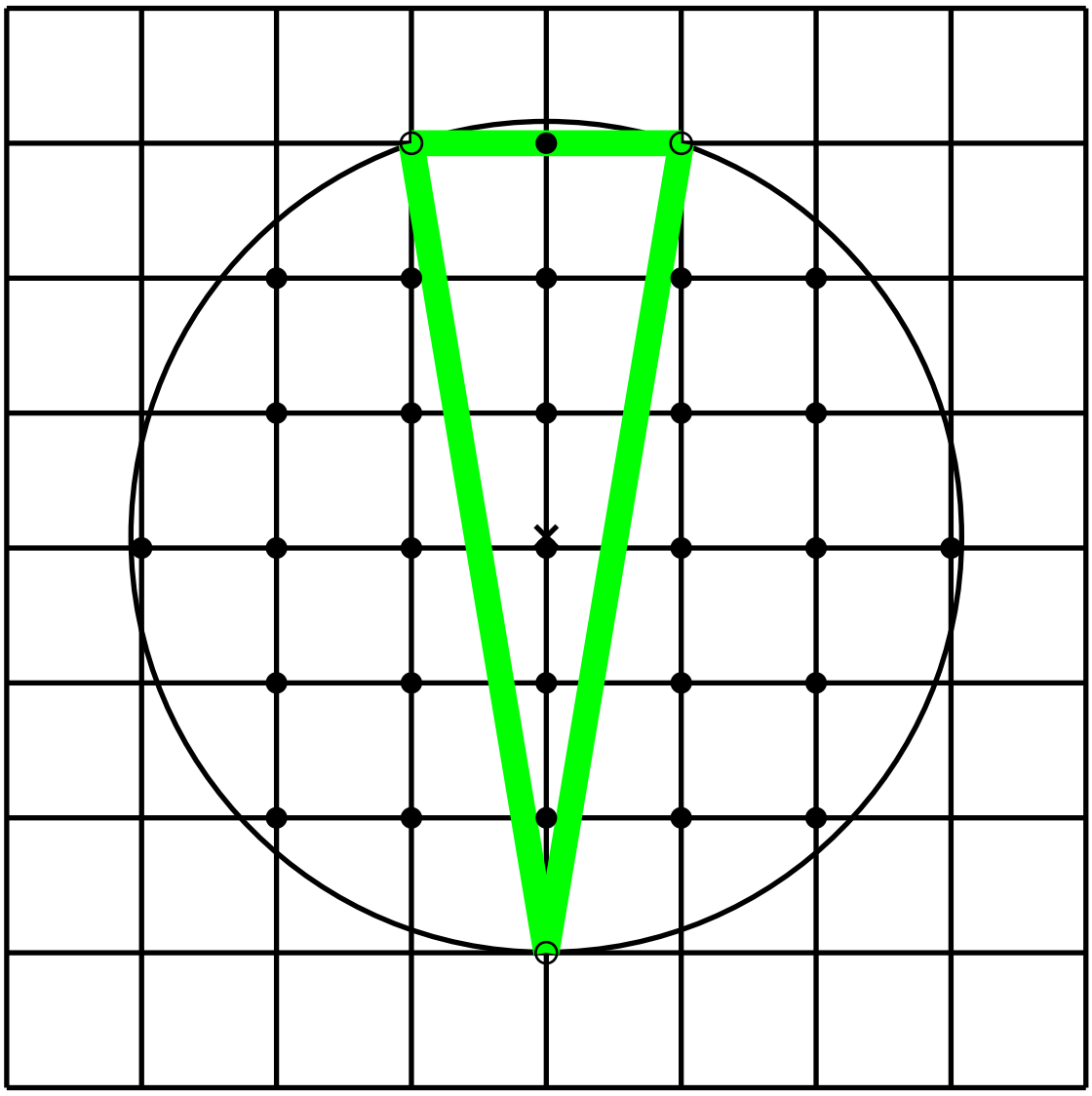
R^2 = 925/98 = 9.43878
R = 3.072259
X = 1/14
Y = 1/14
27 + 4 = 31

A192493(74) = 85, A192494(74) = 9
Triangles: O A



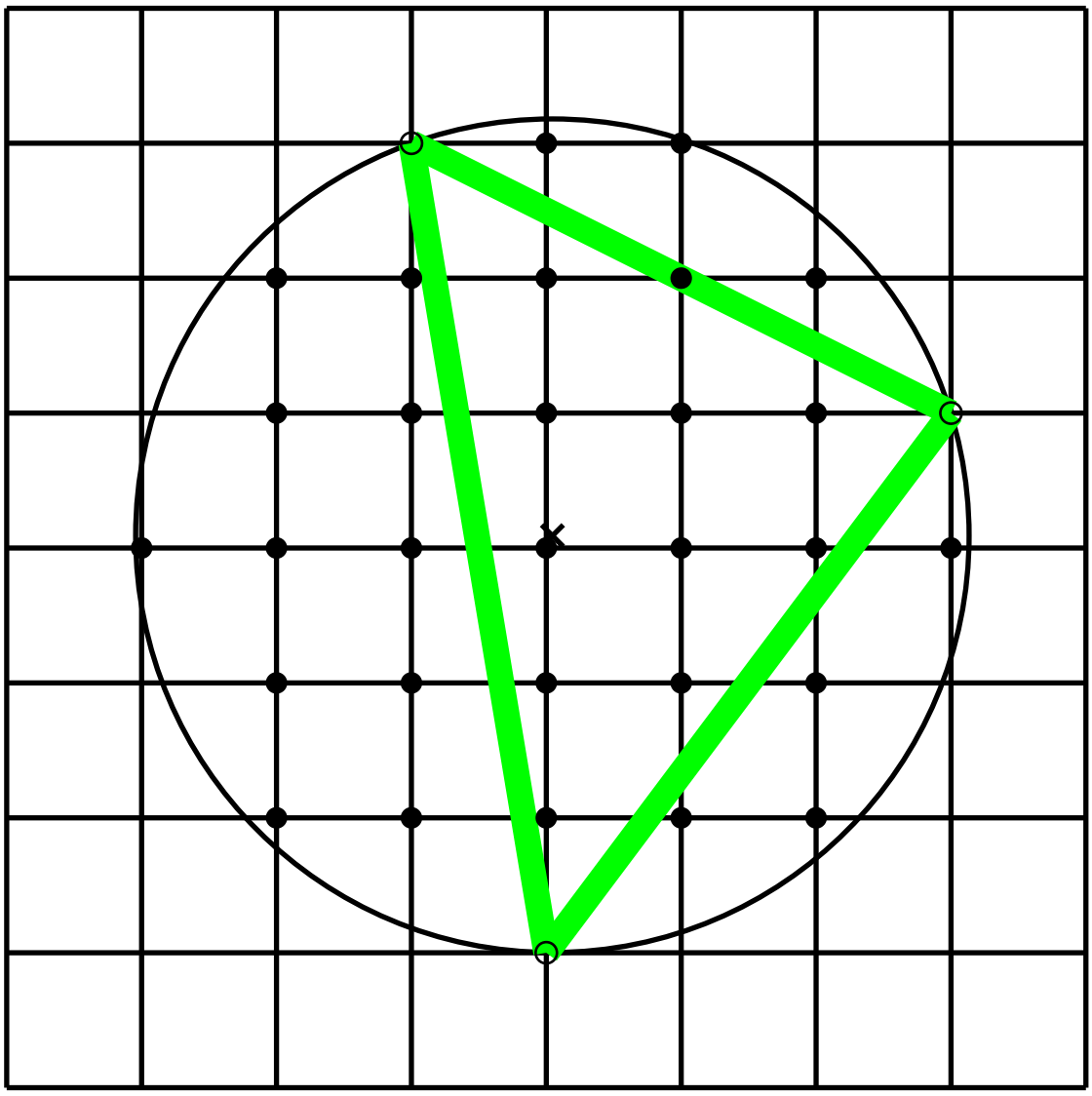
R^2 = 85/9 = 9.44444
R = 3.073181
X = 0/1
Y = 1/3
28 + 4 = 32

A192493(75) = 1369, A192494(75) = 144
Triangles: A



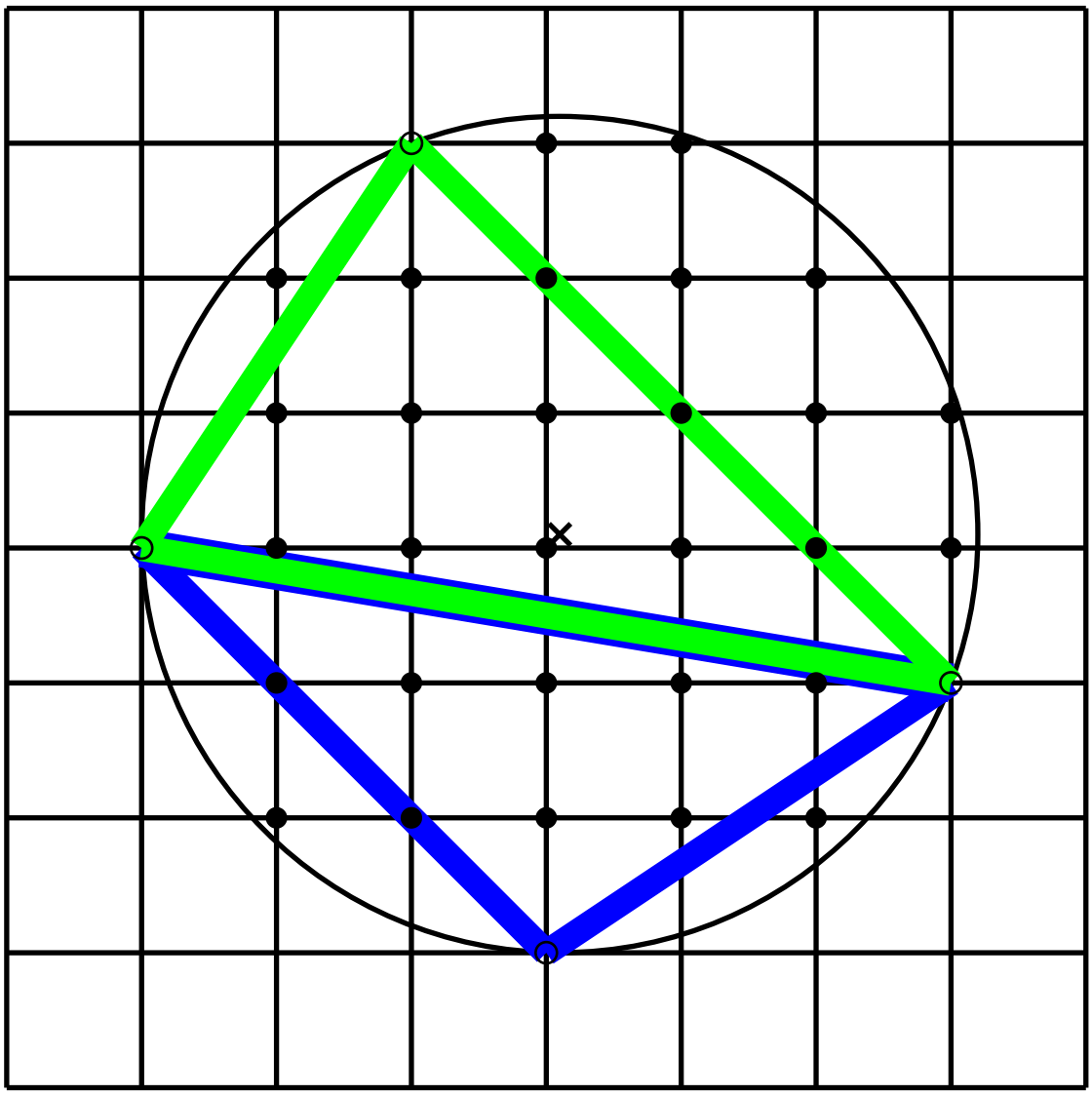
R^2 = 1369/144 = 9.50694
R = 3.083333
X = 0/1
Y = 1/12
28 + 3 = 31

A192493(76) = 4625, A192494(76) = 484
Triangles: A



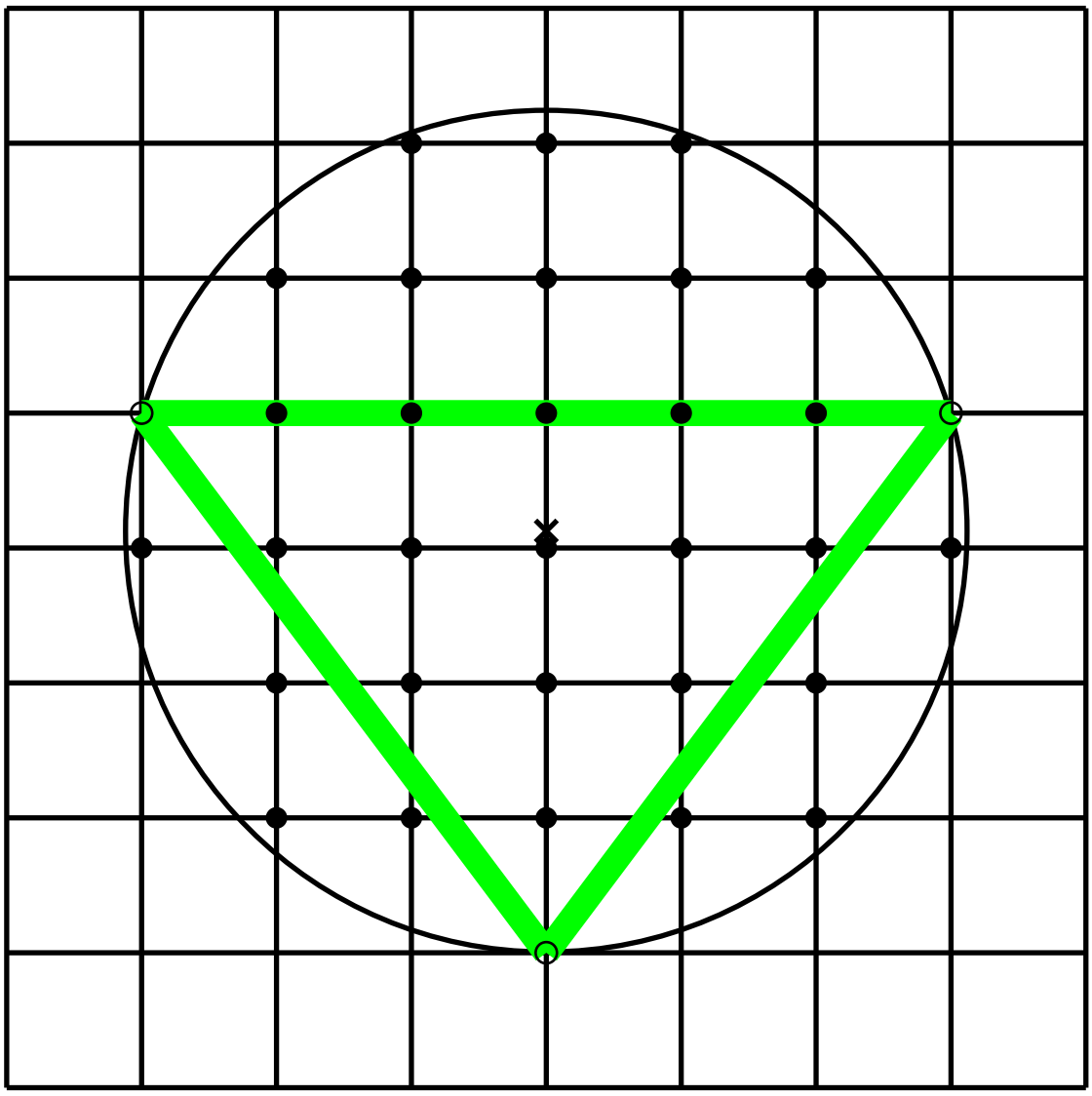
R^2 = 4625/484 = 9.55579
R = 3.091243
X = 1/22
Y = 1/11
29 + 3 = 32

A192493(77) = 481, A192494(77) = 50
Triangles: O A



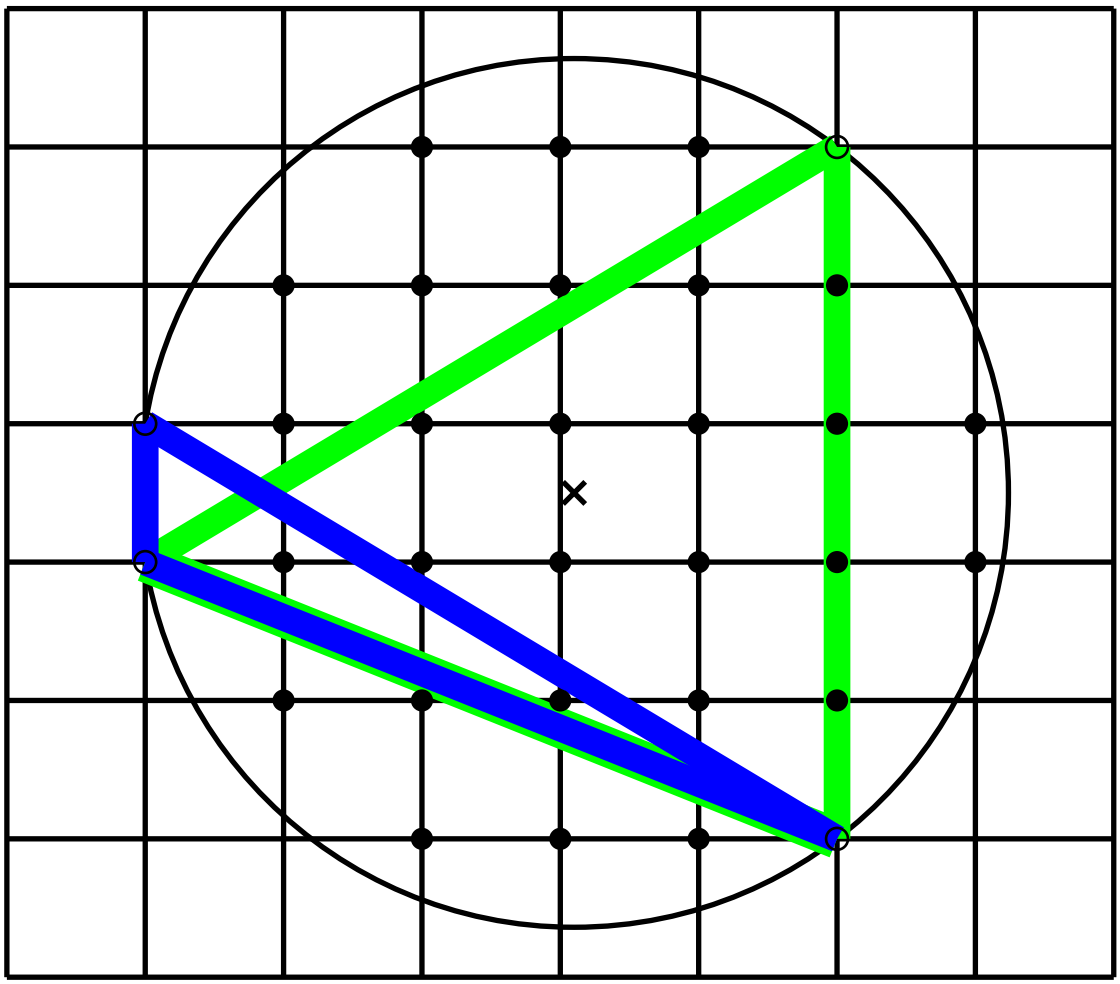
R^2 = 481/50 = 9.62000
R = 3.101612
X = 1/10
Y = 1/10
29 + 4 = 33

A192493(78) = 625, A192494(78) = 64
Triangles: A



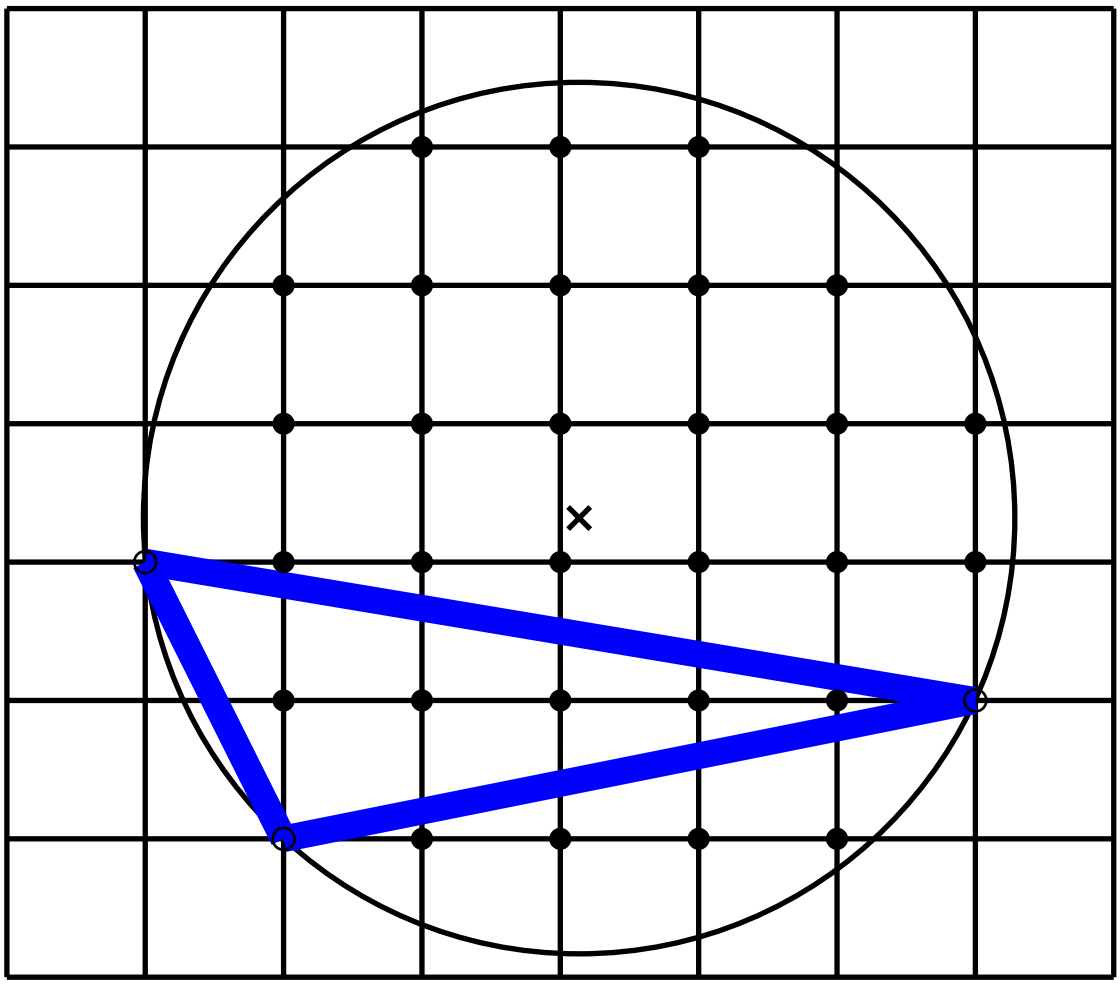
R^2 = 625/64 = 9.76562
R = 3.125000
X = 0/1
Y = 1/8
30 + 3 = 33

A192493(79) = 493, A192494(79) = 50
Triangles: O A



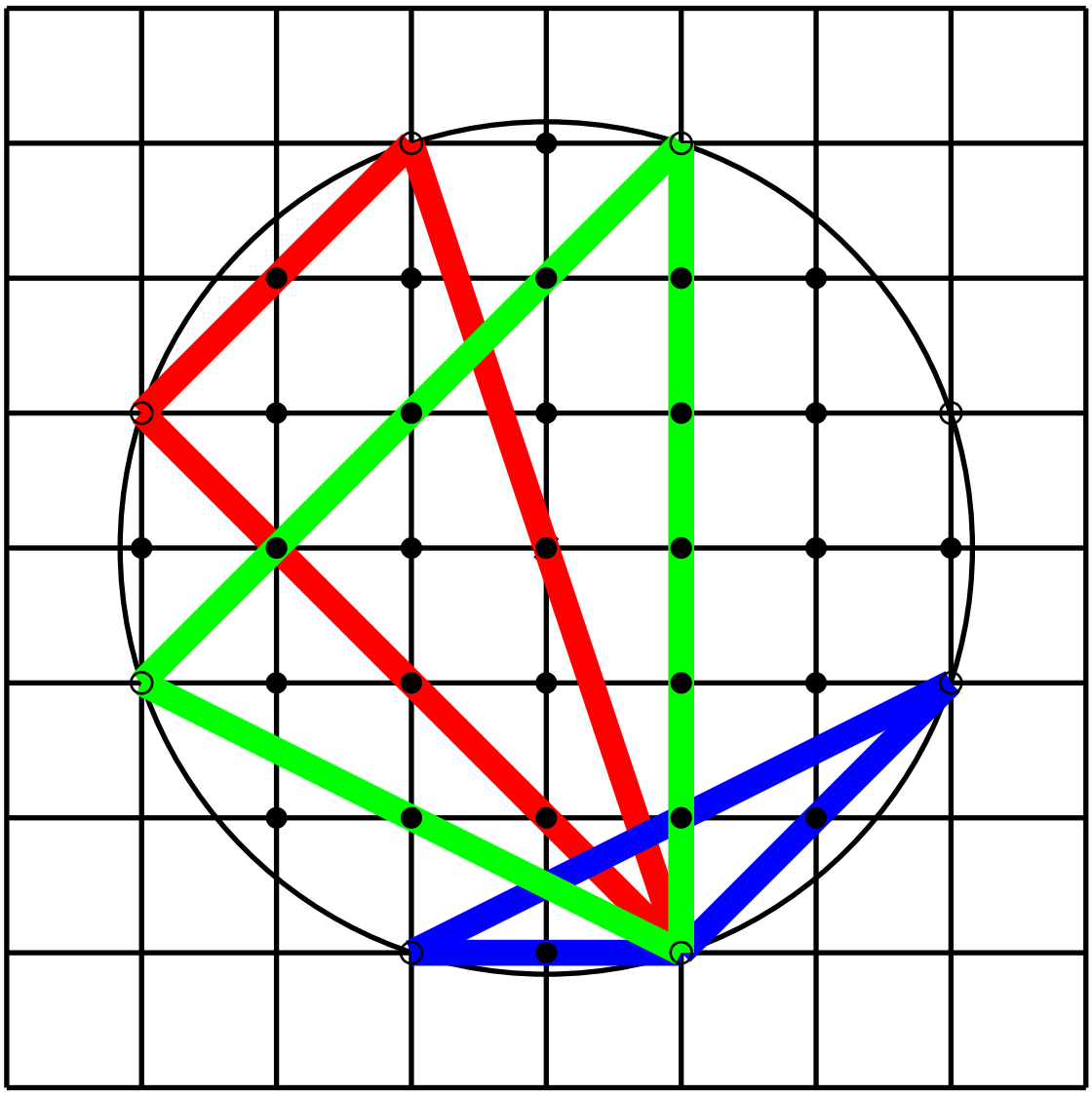
R^2 = 493/50 = 9.86000
R = 3.140064
X = 1/10
Y = 1/2
28 + 4 = 32

A192493(80) = 2405, A192494(80) = 242
Triangles: O



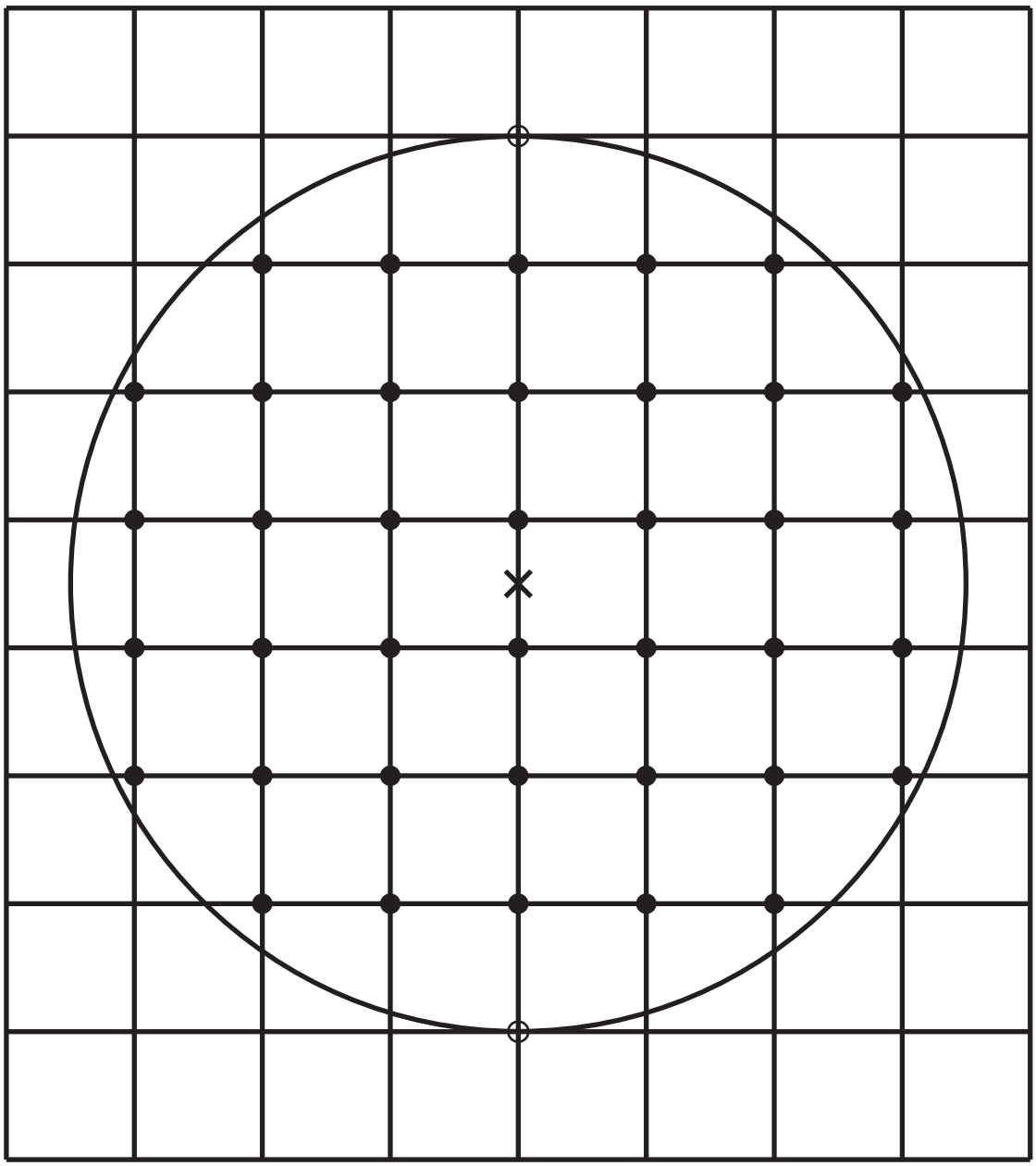
R^2 = 2405/242 = 9.93802
R = 3.152462
X = 3/22
Y = 7/22
29 + 3 = 32

A192493(81) = 10, A192494(81) = 1
Triangles: O R A



R^2 = 10/1 = 10.00000
R = 3.162278
X = 0/1
Y = 0/1
29 + 8 = 37

Special Case, R^2=49/4 not representable
by circumcircle of 3 points of square lattice



$R^2 = 49/4 = 12.25000$

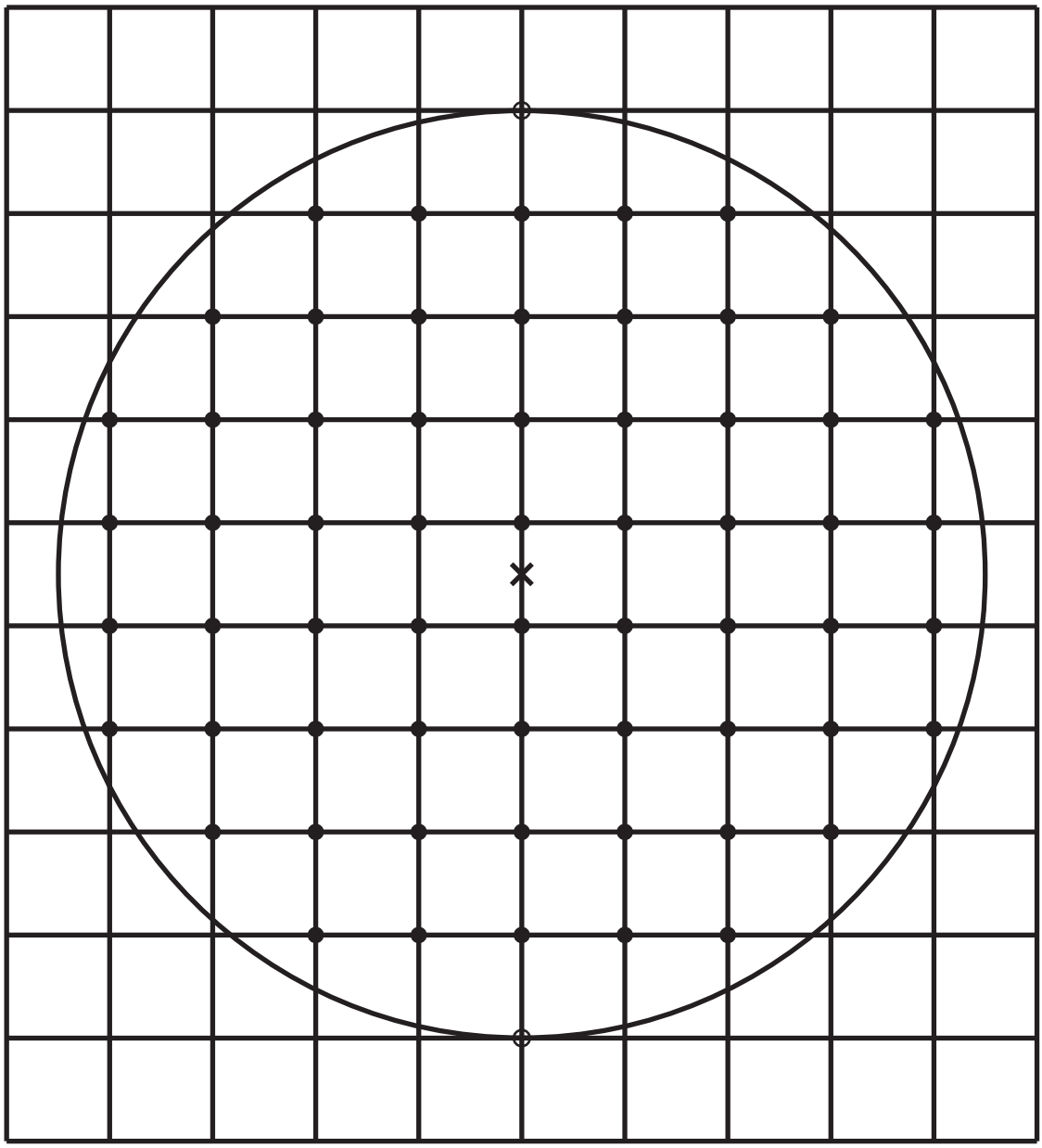
$R = 3.500000$

$X = 0/1$

$Y = 1/2$

$38 + 2 = 40$

Special Case, R^2=81/4 not representable
by circumcircle of 3 points of square lattice



$R^2 = 81/4 = 20.25000$

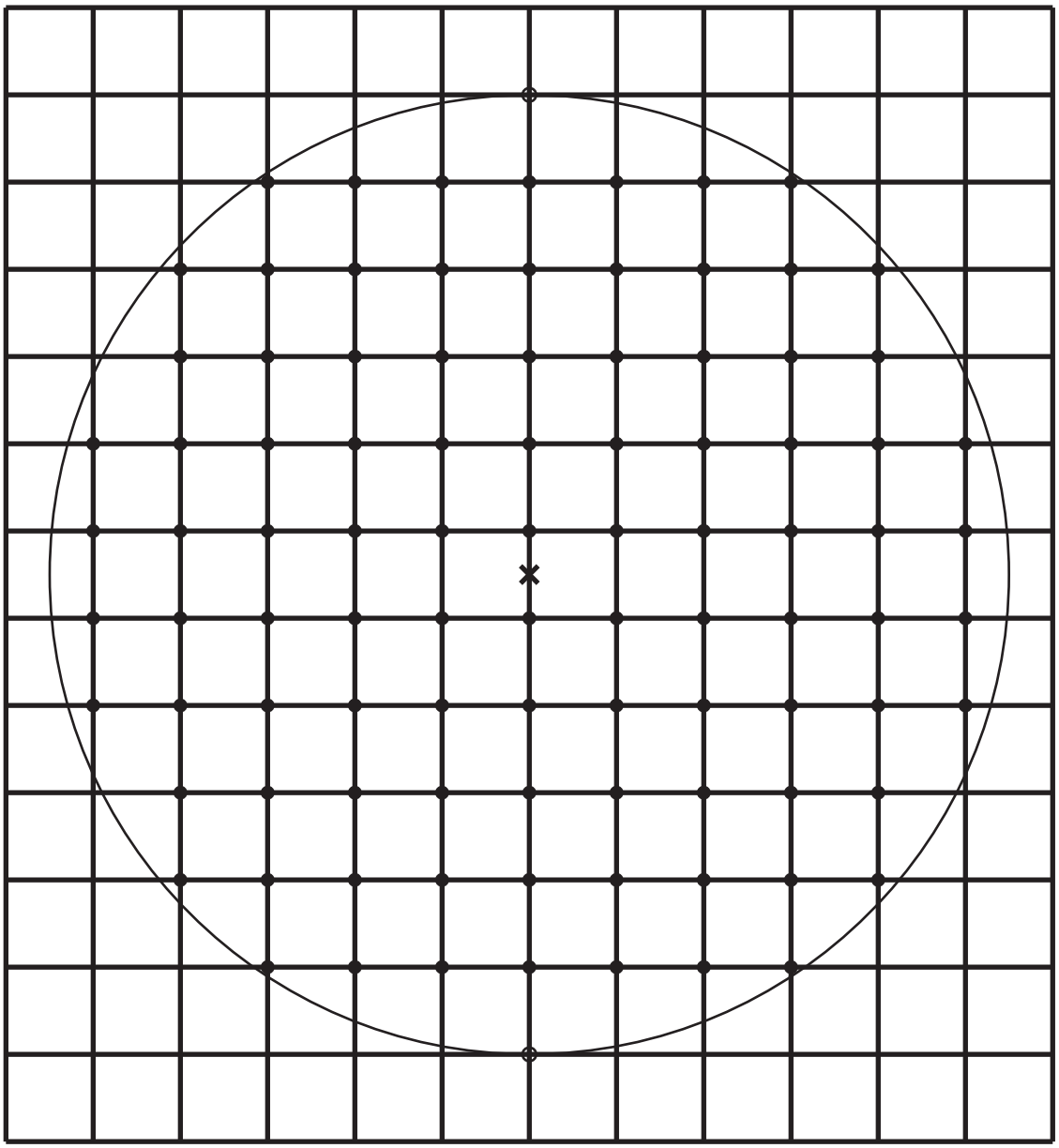
$R = 4.500000$

$X = 0/1$

$Y = 1/2$

$60 + 2 = 62$

Special Case, R^2=121/4 not representable
by circumcircle of 3 points of square lattice



$R^2 = 121/4 = 30.25000$

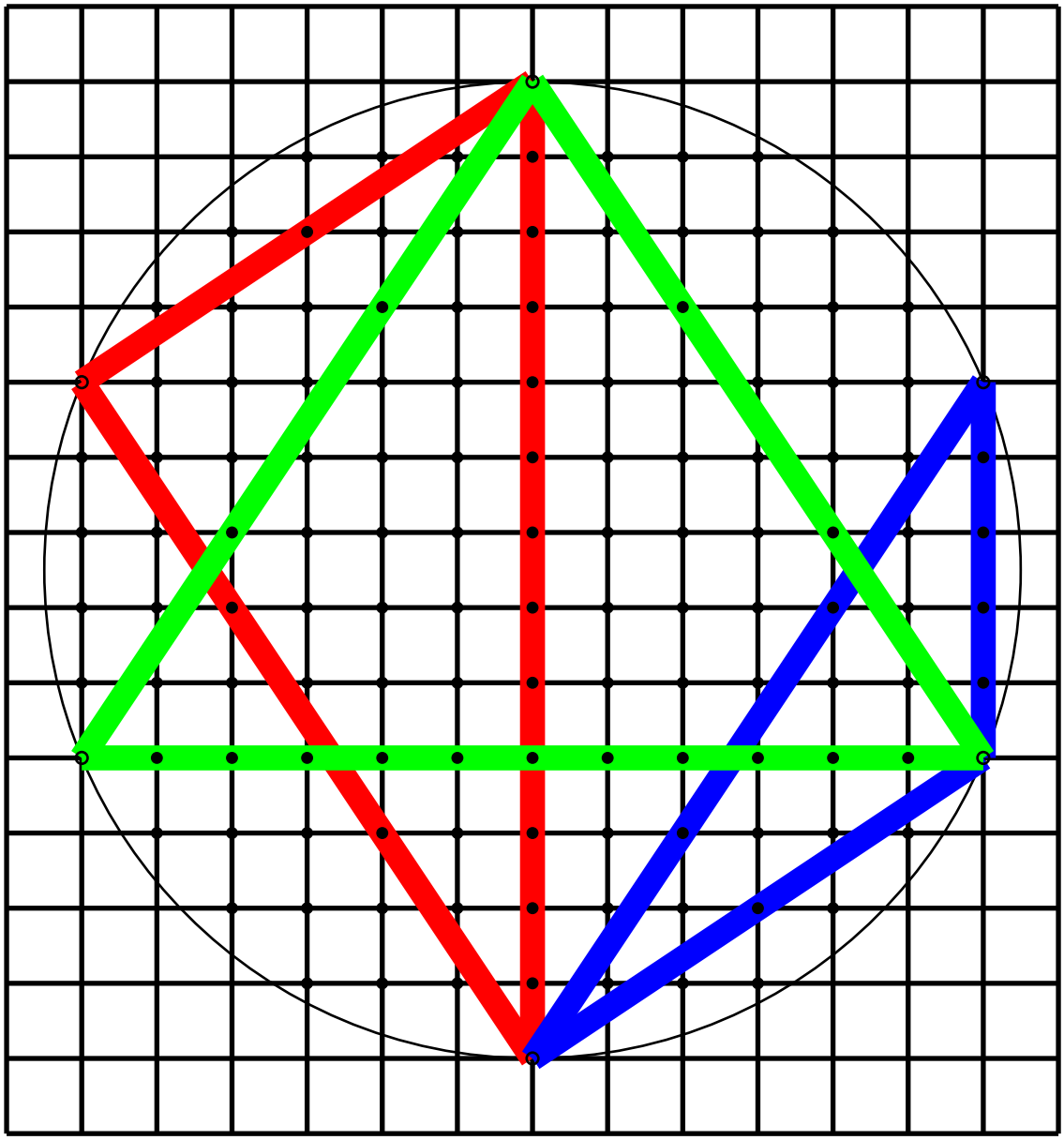
$R = 5.500000$

$X = 0/1$

$Y = 1/2$

$94 + 2 = 96$

In contrast to the special cases R^2 = (1/4) * {1, 9, 49, 81, 121}, R^2 = 169/4 is representable by circumcircles of non-degenerate triangles. The cases in the list are the only minimal enclosing circles of lattice points defined by a diameter with no 3rd point on the circumcircle.



$R^2 = 169/4 = 42.25000$

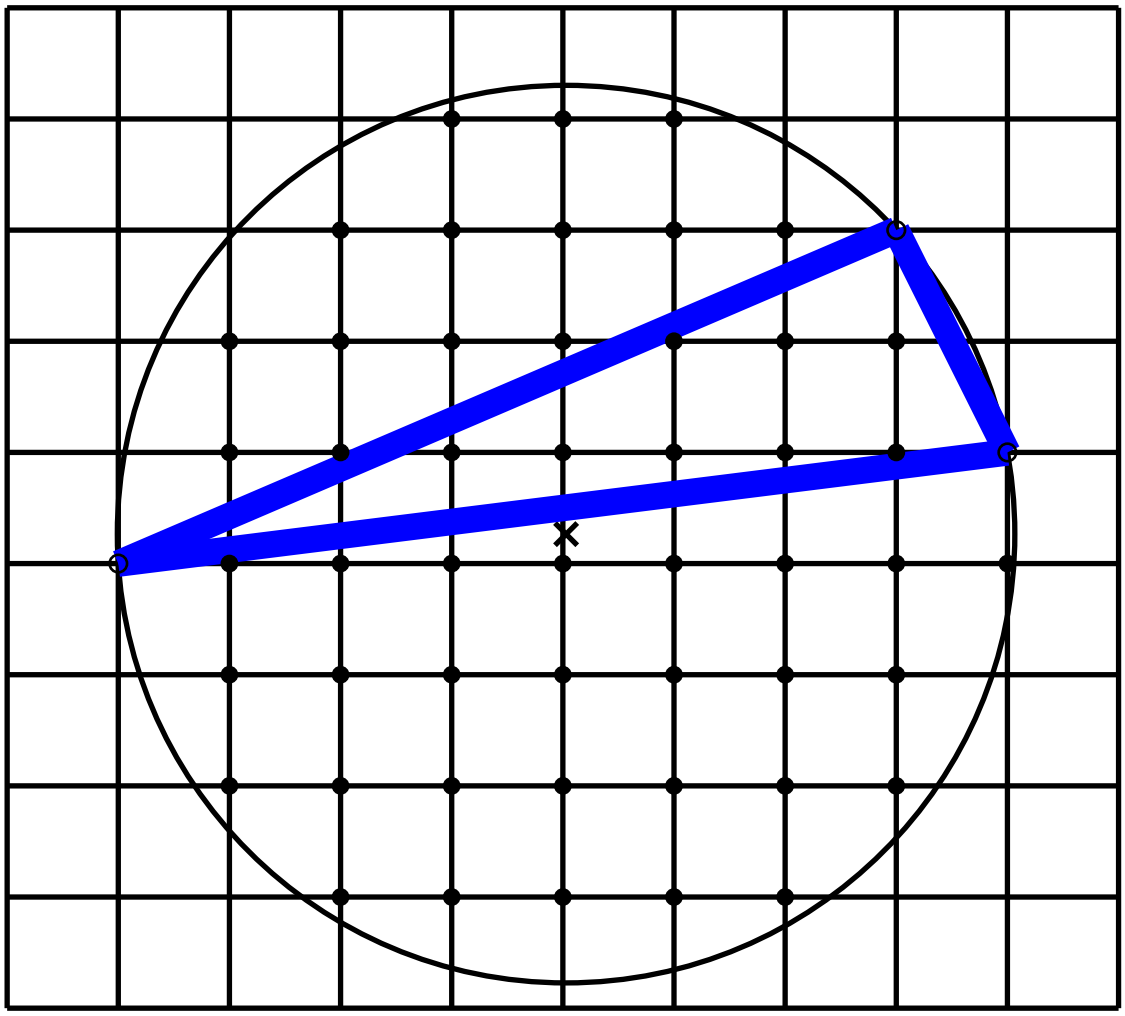
R = 6.500000

X = 0/1

Y = 1/2

128 + 6 = 134

R^2 = 9425 / 578 is the first case with more than one representation, i.e. there are two distinct circles with equal radius but with different center, each of them passing through 3 distinct grid points.
See next page for alternative representation.



$R^2 = 9425/578 = 16.30623$

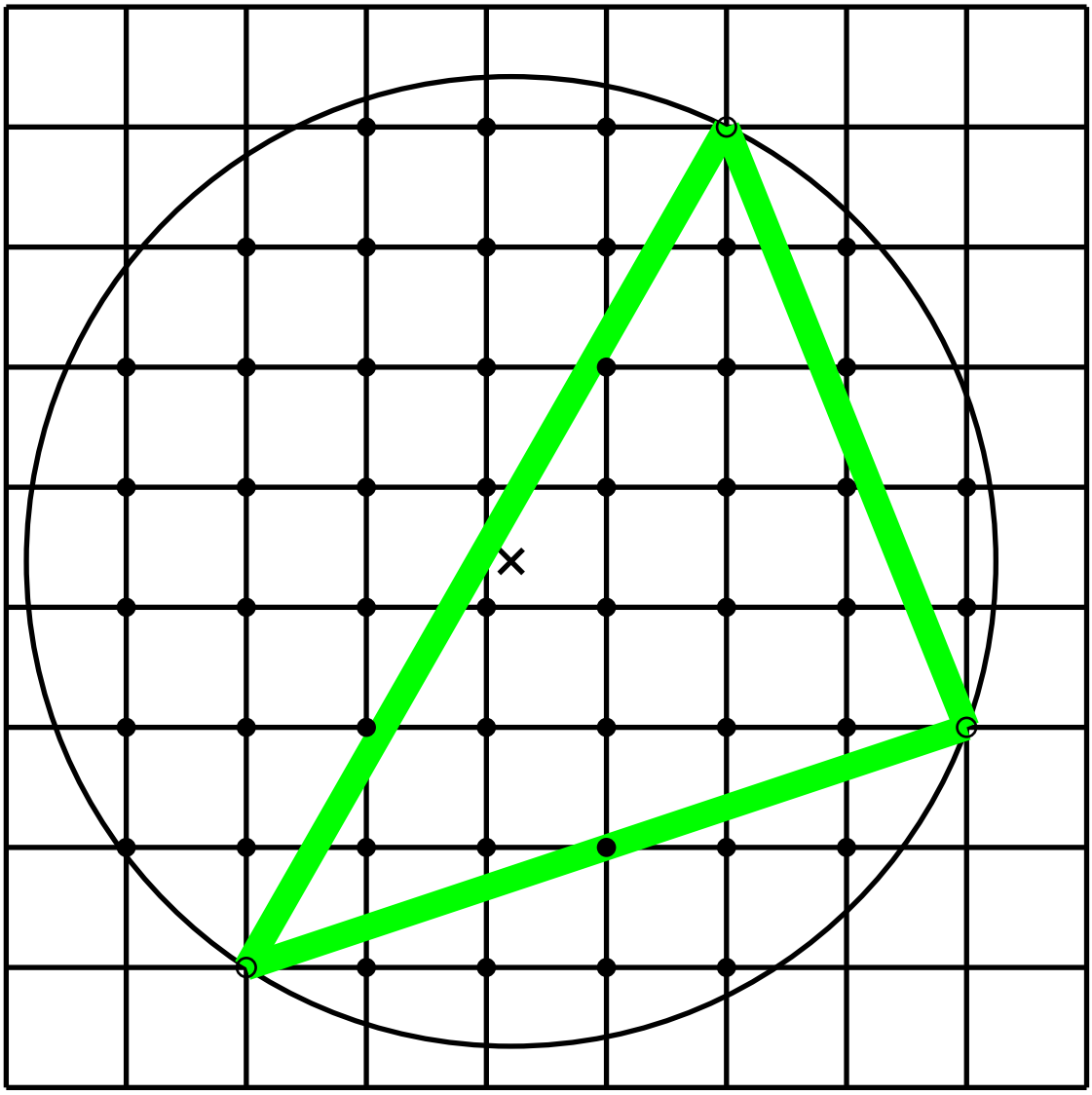
$R = 4.038097$

$X = 1/34$

$Y = 9/34$

$49 + 3 = 52$

Second representation of R^2 = 9425 / 578



R^2 = 9425/ 578 = 16.30623

R = 4 .038097

X = 7 / 34

Y = 13/ 34

50 + 3 = 53