On sequence A192174

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Let $p(0,x)=-1$, $p(1,x)=x$, and for $n\geq 2$ let

$$p(n,x)=xp(n-1,x)-p(n-2,x).$$

Sequence A192174 corresponds to the triangular array $T(n,k)$ defined by

$$T(n,k)=[x^{n-k}]p(n,x), \qquad 0\leq k\leq n.$$

Theorem 1. *Let $n,k\in\mathbb{N}_0$ with $0\leq k\leq n$. Then, if k is odd, then $T(n,k)=0$ and if $k=2j$ is even, then*

$$T(n,2j)=(-1)^j\left(\binom{n-j}{j}-2\binom{n-j-1}{j-1}\right). \tag{1}$$

Proof. Let $U_n(t)$ denote the Chebyshev polynomial of the second kind, defined by $U_0(t)=1, U_1(t)=2t$, and for $n\geq 2$,

$$U_n(t)=2tU_{n-1}(t)-U_{n-2}(t).$$

Set $q_n(x)=U_n(\frac{x}{2})$. Then $q_0(x)=1$, $q_1(x)=x$, and for $n\geq 2$,

$$q_n(x)=xq_{n-1}(x)-q_{n-2}(x).$$

Thus, the q_ns and the $p(n,x)$s satisfy the same recurrence. Now, set $d_n(x)=p(n,x)-q_n(x)$. Then the d_ns also satisfy the same recurrence. We have $d_0(x)=-2$ and $d_1(x)=0$. By induction one easily sees that, $d_{n+2}(x)=2q_n$. It follows that, for $n\geq 2$,

$$p(n,x)=q_n(x)+2q_{n-2}(x)=U_n(\tfrac{x}{2})+2U_{n-2}(\tfrac{x}{2}). \tag{2}$$

Now recall that

$$U_n(\tfrac{x}{2})=\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}x^{n-2j}, \tag{3}$$

Assume that $n \geq 2$. Combining (2) and (3) we immediately see that $T(n,k) = 0$ for odd k. Now, assume that $k = 2j$ is even. Then

$$\begin{aligned} T(n, 2j) &= (-1)^j \binom{n-j}{j} + 2(-1)^{j-1} \binom{n-j-1}{j-1} \\ &= (-1)^j \left(\binom{n-j}{j} - 2\binom{n-j-1}{j-1} \right). \end{aligned}$$

By direct calculation one verifies that the assertion holds for $n = 0$ and $n = 1$. □

The statement of the following corollary was conjectured in A192174.

Corollary 1. *Let $s \in \mathbb{N}$. Then*

$$T(3s, 2s-2) = (-1)^{s+1} \frac{3(2s)!}{(s+2)!(s-1)!} = (-1)^{s+1} \text{A000245}(s).$$

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.