# CHARACTERIZATION OF SOME GOLDEN RATIO SEQUENCES

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Abstract. The integer sequences A190249, A190250 and A190251 of the On-Line Encyclopedia are characterized by conditions on the fractional part of multiples of the golden ratio.

Let $f : n \mapsto f(n)$ denote the sequence A190248 of [1] defined as follows

$$f(n) = \lfloor nu + nv + nw \rfloor - \lfloor nu \rfloor - \lfloor nv \rfloor - \lfloor nw \rfloor$$

where $u = \Phi = (\sqrt{5}+1)/2$ is the golden ratio, $v = \Phi^2$, $w = \Phi^3$, and for $x \in \mathbb{R}$, floor $\lfloor x \rfloor$ is the greatest integer $\leq x$.

The fractional part $\{x\}$ is defined via $\{x\} = x - \lfloor x \rfloor$. For $k \in \mathbb{N}$ it holds $\lfloor k + x \rfloor = k + \lfloor x \rfloor$. Thus, $\lfloor nu + nv + nw \rfloor = \lfloor nu \rfloor + \lfloor nv \rfloor + \lfloor nw \rfloor + \lfloor \{nu\} + \{nv\} + \{nw\} \rfloor$ and, hence,

$$f(n) = \lfloor \{nu\} + \{nv\} + \{nw\} \rfloor.$$

Note that $\Phi^2 = 1 + \Phi$ and $\Phi^3 = \Phi(1+\Phi) = \Phi + (1+\Phi) = 1 + 2\Phi$. Thus, and by $\{k+x\} = \{x\}$,

$$f(n) = \lfloor \{n\Phi\} + \{n\Phi\} + \{n2\Phi\} \rfloor.$$

Moreover, by $\{kx\} = \{k\{x\}\}$,

$$f(n) = \lfloor 2\{n\Phi\} + \{2\{n\Phi\}\} \rfloor \tag{1}$$

As $\{x\} < 1/2$ implies $\{2\{x\}\} = 2\{x\}$, and by (1), it holds

$$\{n\Phi\} < 1/2 \text{ implies } f(n) = \lfloor 4\{n\Phi\} \rfloor \tag{2}$$

As $\{x\} > 1/2$ implies $\{2\{x\}\} = 2\{x\} - 1$, and by (1), it holds

$$\{n\Phi\} > 1/2 \text{ implies } f(n) = \lfloor 4\{n\Phi\} \rfloor - 1 \tag{3}$$

By (2), the 1st and 2nd, and by (3), the 3rd and 4th of the following implications hold

$$\begin{aligned} \{n\Phi\} < 1/4 \quad &\Rightarrow \quad f(n) = 0, \\ 1/4 < \{n\Phi\} < 1/2 \quad &\Rightarrow \quad f(n) = 1, \\ 1/2 < \{n\Phi\} < 3/4 \quad &\Rightarrow \quad f(n) = 1, \\ \{n\Phi\} > 3/4 \quad &\Rightarrow \quad f(n) = 2. \end{aligned}$$

The sequences A190249, A190250, A190251, resp., are defined as positions of 0, 1, 2, resp., in A190248. We conclude that

$$\begin{aligned} \{n\Phi\} < 1/4 \quad &\text{characterises A190249}, \\ 1/4 < \{n\Phi\} < 3/4 \quad &\text{characterises A190250}, \\ \{n\Phi\} > 3/4 \quad &\text{characterises A190251}. \end{aligned}$$

## References

[1] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A190248.

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