Maple-assisted proof of recurrence for A189619

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There are $2^8 = 256$ configurations for a 4×2 sub-array.

We encode these configurations as lists in the order $\begin{bmatrix} x_1 & x_5 \\ x_2 & x_6 \\ x_3 & x_7 \\ x_4 & x_8 \end{bmatrix}$ and enumerate them.

```
> Configs:= [seq(convert(n,base,2)[1..8],n=2^8..2^9-1)]:
```

Consider the 256×256 transition matrix T with entries $T_{ij} = 1$ if the first two columns of a 4×3 sub-array could be in configuration i while the last two columns are in configuration j, and 0 otherwise. The following code computes it.

```
> Compatible:= proc(i,j) local Xi,Xj,k;
   Xi:= Configs[i]; Xj:= Configs[j];
   if Xi[5..8] <> Xj[1..4] then return 0 fi;
   if Xi[1]=0 and ((Xi[5]=1 and Xj[5]=0) or (Xi[6]=1 and Xj[7]=0))
  then return 0 fi;
   if Xi[2]=0 and ((Xi[6]=1 and Xj[6]=0) or (Xi[7]=1 and Xj[8]=0))
  then return 0 fi;
   if Xi[3]=0 and ((Xi[6]=1 and Xj[5]=0) or (Xi[7]=1 and Xj[7]=0))
  then return 0 fi;
   if Xi[4]=0 and ((Xi[7]=1 and Xj[6]=0) or (Xi[8]=1 and Xj[8]=0))
  then return 0 fi;
   1
  end proc:
  T:= Matrix(256,256,Compatible):
```

Thus for $n \geq 2$ $a(n) = u^T T^{n-2} u$ where u is a column vector of all 1's.

```
> u:= Vector(256,1):
```

To check, here are the first few entries of our sequence (including a_1, which doesn't really fit the pattern, although it will work with the recurrence). For future use, we pre-compute more $T^n v$ than we need.

```
> Tu[0]:= u:
  for nn from 1 to 60 do Tu[nn]:= T . Tu[nn-1] od:
> A:= [16, seq(u^%T . Tu[n],n=0..50)]:
  A[1..20];
```

[16, 256, 1723, 8496, 50024, 357323, 2482591, 15915001, 100745265, 655633882, 4322765564, 28292943180, 183873191611, 1195974189947, 7802581300173, 50931021729006, 332106212299242, 2164490825527781, 14110729853099870, 92015347002838934] **(1)**

This is the empirical recurrence:

```
> empirical:= a(n) = 8*a(n-1) -14*a(n-2) +10*a(n-3) +258*a(n-4)
```

```
-737*a(n-5) -1502*a(n-6) +1780*a(n-7) +8033*a(n-8) -4800*a(n-9)
-30115*a(n-10) +54016*a(n-11) -79554*a(n-12) -215482*a(n-13)
+787373*a(n-14) -626468*a(n-15) -426934*a(n-16) +3545678*a(n-17)
-6093224*a(n-18) +8409993*a(n-19) -6556066*a(n-20) +986540*a
(n-21) -4549030*a(n-22) +9400135*a(n-23) -23776243*a(n-24)
+29721357*a(n-25) -36187856*a(n-26) +30281962*a(n-27) -31933555*a
(n-28) +10845312*a(n-29) +5786415*a(n-30) +7694508*a(n-31)
+5236176*a(n-32) -9921669*a(n-33) -713908*a(n-34) +1520657*a
(n-35) -3672439*a(n-36) +1127159*a(n-37) +4759522*a(n-38)
-673175*a(n-39) -2818276*a(n-40) -151046*a(n-41) +1043153*a(n-42)
+59137*a(n-43) -154754*a(n-44) -24424*a(n-45) +16833*a(n-46)
+160*a(n-47) -631*a(n-48) -60*a(n-49) +4*a(n-50):
```

This says that $u^T T^{n-2} P(T)\, u = 0$ where $P(x)$ is the following polynomial.

```
> P:= x^50 - add(coeff(rhs(empirical),a(n-j))*x^(50-j),j=1..50);
```

$$P := x^{50} - 8x^{49} + 14x^{48} - 10x^{47} - 258x^{46} + 737x^{45} + 1502x^{44} - 1780x^{43} - 8033x^{42} + 4800x^{41} + 30115x^{40} - 54016x^{39} + 79554x^{38} + 215482x^{37} - 787373x^{36} + 626468x^{35} + 426934x^{34} - 3545678x^{33} + 6093224x^{32} - 8409993x^{31} + 6556066x^{30} - 986540x^{29} + 4549030x^{28} - 9400135x^{27} + 23776243x^{26} - 29721357x^{25} + 36187856x^{24} - 30281962x^{23} + 31933555x^{22} - 10845312x^{21} - 5786415x^{20} - 7694508x^{19} - 5236176x^{18} + 9921669x^{17} + 713908x^{16} - 1520657x^{15} + 3672439x^{14} - 1127159x^{13} - 4759522x^{12} + 673175x^{11} + 2818276x^{10} + 151046x^{9} - 1043153x^{8} - 59137x^{7} + 154754x^{6} + 24424x^{5} - 16833x^{4} - 160x^{3} + 631x^{2} + 60x - 4 \quad (2)$$

It turns out that $P(T)\, u = 0$. Verifying this completes the proof.

```
> R:= add(coeff(P,x,j)*Tu[j],j=0..50):
> R^%T . R;
```

$$0 \quad (3)$$