On the coefficients of $(r+\sqrt{p}+\sqrt{q})^n$

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Let r be an integer and let $p,q>1$ be two coprime natural numbers. Let $(a_n),(b_n),(c_n)$, and (d_n) be the sequences corresponding to the coeffients of $1,\sqrt{p},\sqrt{q}$, and $\sqrt{pq}$, respectively, in the expansion of $(r+\sqrt{p}+\sqrt{q})^n$. Let $A(x),B(x),C(x)$, and $D(x)$ be the corresponding generating functions. We have

$$\begin{aligned}
&a_{n+1}+b_{n+1}\sqrt{p}+c_{n+1}\sqrt{q}+d_{n+1}\sqrt{pq}\\
=&(r+\sqrt{p}+\sqrt{q})^{n+1}\\
=&(a_n+b_n\sqrt{p}+c_n\sqrt{q}+d_n\sqrt{pq})(r+\sqrt{p}+\sqrt{q})\\
=&ra_n+pb_n+qc_n+(a_n+rb_n+qd_n)\sqrt{p}+(a_n+rc_n+pd_n)\sqrt{q}+(b_n+c_n+rd_n)\sqrt{pq}
\end{aligned}$$

It follows that

$$\begin{aligned}
A(x)&=1+x(rA(x)+pB(x)+qC(x))\\
B(x)&=x(A(x)+rB(x)+qD(x))\\
C(x)&=x(A(x)+rC(x)+pD(x))\\
D(x)&=x(B(x)+C(x)+rD(x))
\end{aligned}$$

Thus, in matrix notation, we wish to solve the system $Mv=w$, where

$$M=\begin{pmatrix}1-rx & -px & -qx & 0\\ -x & 1-rx & 0 & -qx\\ -x & 0 & 1-rx & -px\\ 0 & -x & -x & 1-rx\end{pmatrix}, v=\begin{pmatrix}A(x)\\B(x)\\C(x)\\D(x)\end{pmatrix}, w=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

With

$$\phi=(r^4-2r^2(p+q)+(p-q)^2)x^4+4r(p+q-r^2)x^3+2(3r^2-p-q)x^2-4rx+1$$

We then have

$$A(x) = \frac{r(p+q-r^2)x^3 + (3r^2-p-q) - 3rx + 1}{\phi}$$

$$B(x) = \frac{x((r^2-p+q) - 2rx + 1)}{\phi}$$

$$C(x) = \frac{x((r^2+p-q) - 2rx + 1)}{\phi}$$

$$D(x) = \frac{2x^2(-rx+1)}{\phi}$$

Example

Taking $r = 1, p = 2$, and $q = 3$, we confirm the conjectures stated in A188571, A188572, and A188573.