Division gets rough: OEIS A187824 and A220890

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2020 March 14

1 Introduction

From the OEIS[1]:

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%I A187824
%S 3,4,5,6,3,4,4,5,3,6,4,4,3,5,5,4,3,6,5,5
%N a(n) is the largest m such that n is
   congruent to -1, 0 or 1 mod k for all
   k from 1 to m.
%K nonn,nice
%O 2,1
%A Kival Ngaokrajang, Dec 27 2012

%I A220890
%S -1,-1,-1,2,3,4,5,29,41,55,71,881,791,9360
%N a(n) = least m such that A187824(m) = n,
   or -1 if A187824 never takes the value n.
%Y Cf. A187824, A056697, A220891.
%K sign
%O 0,4
%A N. J. A. Sloane, Dec 30 2012
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These sequences are about “rough division:” m roughly divides n iff m divides any of $n-1, n, n+1$. A187824$(n)+1$ is the least rough non-divisor of n. Obviously, 1, 2, and 3 roughly divide each integer, so $a(n)+1 > 3$.

1.1 Terminology

- Δ, Δ_i: a number in $\{-1, 0, 1\}$.
- $\not\approx$ (roughly divides): $a \not\approx b$ iff $\exists k, \Delta : b = ka + \Delta$.
- $srnd(n)$: the smallest rough non-divisor of n, A187824$(n)+1$.

A220890$(n-1)$ is the least inverse of $srnd$. Of course it is undefined at 1, 2, and 3; but there are other non-values:
$\forall x : srnd(x) \notin \{1, 2, 3, 18, 20, 24, ...\}$. (Proof later.)

2 Some theorems

Theorem 1 If $a|b$ and $b \not\approx n$, then $a \not\approx n$.
Proof:
$b \not\approx n$, so $\exists x, \Delta : n + \Delta = bx$
$a|b$, so $\exists y : b = ya$
$n + \Delta = yax = yx \cdot a$
$a \not\approx n$ ▪

Theorem 2 If $a \not\approx n$ and $b \not\approx n$ and $\gcd(a, b) > 2$, then $\operatorname{lcm}(a, b) \not\approx n$.
Proof:
Let $g = \gcd(a, b)$; $a' = a/g$; $b' = b/g$.
$a \not\approx n$, so $\exists x, \Delta_1 : n + \Delta_1 = ax$
$b \not\approx n$, so $\exists y, \Delta_2 : n + \Delta_2 = by$
$\Delta_1 - \Delta_2 = ax - by$
$g|(ax - by)$ (because $g|a$ and $g|b$)
$g|(\Delta_1 - \Delta_2)$
$0 = \Delta_1 - \Delta_2$ (because $|\Delta_1 - \Delta_2| \leq 2 < g$)
$0 = ax - by = a'x - b'y$
$b'|(a'x)$
$b'|x$ (because $\gcd(a', b') = 1$)
$\exists z : x = zb'$
$n + \Delta_1 = azb' = a'gb'z = \operatorname{lcm}(a, b) \cdot z$
$\operatorname{lcm}(a, b) \not\approx n$ ▪

Theorem 3 If $\gcd(a, b) > 2$, $v|\operatorname{lcm}(a, b)$, $v > a$, and $v > b$, then v is never a smallest rough non-divisor.
Proof:
Let $m = \operatorname{lcm}(a, b)$. For each n:

- If $a \not\approx n$ and $b \not\approx n$, then theorem 2 applies, and $m \not\approx n$. For each suitable v, $v|m$, and so $v \not\approx n$ (theorem 1). v is not a rough non-divisor.
- Otherwise, a or b is a rough non-divisor, less than any suitable v, and so v is not the smallest rough non-divisor. ▪

Examples from M.F.Hasler’s comments in A220890:

$a = 9$, $b = 12$: gcd $= 3$, lcm $= 36$
If $9 \not\approx n$ and $12 \not\approx n$, then $36 \not\approx n$.
Therefore the factors of 36 greater than 12: 18, 36 are never $srnd(x)$,

$a = 12$, $b = 15$: $\gcd = 3$, $\mathrm{lcm} = 60$
If $12 \mathrel{\approx\!\!\!\!\mid} n$ and $15 \mathrel{\approx\!\!\!\!\mid} n$, then $60 \mathrel{\approx\!\!\!\!\mid} n$,
20, 30, 60 are never $srnd(x)$.

$a = 8$, $b = 12$: $\gcd = 4$, $\mathrm{lcm} = 24$
If $8 \mathrel{\approx\!\!\!\!\mid} n$ and $12 \mathrel{\approx\!\!\!\!\mid} n$, then $24 \mathrel{\approx\!\!\!\!\mid} n$.
24 is never $srnd(x)$.

Theorem 4 Let $b > 3$ be a non-divisor of m. Then there is an integer x such that $m \mathrel{\approx\!\!\!\!\mid} x$ but not $b \mathrel{\approx\!\!\!\!\mid} x$.
Proof:
Let $g = \gcd(b, m)$.
$\exists x, y : bx + my = g$
not $b|m$, so $g < b$
$\exists \Delta : \Delta + g \notin \{-1, 0, 1\} \bmod b$
  ($\Delta = \max(2 - g, -1)$ works.)
Let $x = \Delta + my = \Delta + g - bx$.
$m \mathrel{\approx\!\!\!\!\mid} x$ but not $b \mathrel{\approx\!\!\!\!\mid} x$ ∎

3 The range of $srnd(x)$

Theorem 3 yields many non-*srnd* values.

- For primes p, q, r with $p < q < r$:
  If $pqr|n$, let $a = n/p$, $b = n/q$. Then
  $r|\gcd(a, b) > 2$, $b < a < n$, $n|n = \mathrm{lcm}(a, b)$.
- For primes p, q with $3 < p < q$:
  If $pq|n$, let $a = 3n/p$, $b = 3n/q$. Then
  $3|\gcd(a, b) > 2$, $b < a < n$, $n|3n = \mathrm{lcm}(a, b)$.
- For primes p, q with $q > 2$ and $p \neq q$:
  If $pq^2|n$, let $a = n/p$, $b = n/q$. Then
  $q|\gcd(a, b) > 2$, $a < n$, $b < n$, $n|n = \mathrm{lcm}(a, b)$.
- For prime $p > 3$:
  If $4p|n$, let $a = 3n/4$, $b = 3n/p$. Then
  $3|\gcd(a, b) > 2$, $b < a < n$, $n|3n = \mathrm{lcm}(a, b)$.
- If $24|n$, let $a = n/4$, $b = n/3$. Then
  $4|\gcd(a, b) > 2$, $a < b < n$, $n|n = \mathrm{lcm}(a, b)$.

In each case, theorem 3 applies, and n is not a *srnd* value.

The non-*srnd* values include all multiples of

- pqr (three distinct primes),
- pq if $3 < p < q$,
- pq^2 if $2 < q$,
- $2^2 q$ if $3 < q$,
- $2^3 \cdot 3$.

That leaves 1, p, $2p$, $3p$, p^k, and $2^2 \cdot 3 = 12$.

$1, 2, 3, 12$

1, 2 and 3 roughly divide anything, so are not *srnd* values. $srnd(881) = 12$, so 12 is a *srnd* value.

$p^k > 3$

For prime p, let $b = p^k > 3$. (If $p > 3$ then $k \geq 1$; else $k > 1$.)
Let $a_i = i$, for each $1 \leq i < b$; let $m = \mathrm{lcm}(a_i)$.
Then not $b|m$, and by theorem 4, there is an x such that $m \mathrel{\approx\!\!\!\!\mid} x$, and so all $a_i \mathrel{\approx\!\!\!\!\mid} x$, but not $b \mathrel{\approx\!\!\!\!\mid} x$.
Therefore b is a *srnd* value.

$2p$

For odd prime p, let a_i be the values from 1 to $2p - 1$, except for p. Let $m = \mathrm{lcm}(a_i)$; then m is even and $\gcd(p, m) = 1$.
$\exists x, y : px + my = 1$; x is odd
Let $n = px = 1 - my$.
for each a_i, $a_i \mathrel{\approx\!\!\!\!\mid} n$
$n = px \equiv p \bmod 2p$ (because x is odd)
$p \mathrel{\approx\!\!\!\!\mid} n$ but not $2p \mathrel{\approx\!\!\!\!\mid} n$

And so $srnd(n) = 2p$. Also, $srnd(2) = 4$; so each $2p$ is a *snrd* value.

$3p$

For prime $p > 3$, let a_i be the values from 1 to $3p - 1$, except for p and $2p$. Let $m = \mathrm{lcm}(a_i)$; then $3|m$ and $\gcd(p, m) = 1$.
$\exists x, y : px + my = 1$; not $3|x$
Let $n = 2px - 1 = 1 - 2my$.
for each a_i, $a_i \mathrel{\approx\!\!\!\!\mid} n$; $p \mathrel{\approx\!\!\!\!\mid} n$ and $2p \mathrel{\approx\!\!\!\!\mid} n$
$3p|3px = n + 1 + px$, $n \equiv -1 - px \bmod 3p \not\equiv \Delta$ (because not $3|x$)
not $3p \mathrel{\approx\!\!\!\!\mid} n$

And so $srnd(n) = 3p$. Also, $srnd(4) = 6$, $srnd(41) = 9$; so each $3p$ is a *snrd* value.

A220890

And so the factorizations of $n + 1$ reveal the holes of A220890(n). If $n + 1 = p^k > 3$, $n + 1 = 2p$, $n + 1 = 3p$, or $n + 1 = 12$, the sequence has a value; otherwise -1.

References

[1] Neil Sloane, *The Online Encyclopedia of Integer Sequences,* `http://oeis.org`