Maple-assisted derivation of recurrence for A185794

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31 Oct 2019

There are $2^{12} = 2048$ configurations for a 2×6 sub-array, but not all can arise: we need no 2×2 subblock sum equal to that of a horizontal neighbour.

We encode these configurations as lists in the order $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{bmatrix}$ and enumerate them. It turns out that there are 676 possible configurations.

```
> Configs:= select(t -> t[1]+t[7] <> t[3]+t[9] and t[2]+t[8] <> t
  [4]+t[10] and t[3]+t[9] <> t[5]+t[11] and t[4]+t[10]<>t[6]+t[12],
  [seq(convert(n,base,2)[1..12],n=2^12..2^13-1)]): nops(Configs);
```

$$676 \tag{1}$$

Consider the 676×676 transition matrix T with entries $T_{ij} = 1$ if the first two rows of a 3×5 sub-array could be in configuration i while the last two rows are in configuration j, and 0 otherwise. The following code computes it.

```
> Compatible:= proc(i,j) local Xi,Xj,k;
   Xi:= Configs[i]; Xj:= Configs[j];
   if Xi[7..12] <> Xj[1..6] then return 0 fi;
   for k from 1 to 5 do if Xi[k]+Xi[k+1] = Xj[k+6]+Xj[k+7] then
  return 0 fi od;
   1
  end proc:
  T:= Matrix(676,676,Compatible):
```

Then $a(n) = u^T\, T^{n-1} u$ where u is the all-ones column vector of length 676.

```
> u:= Vector(676, 1):
```

To check, here are the first few entries of our sequence (including a_1, which doesn't really fit the pattern). For future use, we pre-compute more $T^n u$ than we need.

```
> Tu[0]:= u:
  for nn from 1 to 150 do Tu[nn]:= T . Tu[nn-1] od:
> A:= [seq(u^%T . Tu[n],n=0..150)]:
  A[1..23];
```

$$[676, 1408, 3196, 9368, 23064, 61676, 149828, 433296, 1073036, 3000848, 7423780, 21476064, 53645816, 153197812, 383014100, 1109241856, 2792472812, 8043408452, 20257150840, 58652173360, 148613711308, 429039760096, 1086671505564] \tag{2}$$

The recurrence shows up as a linear dependence among $T^n u$. We gather these as columns of a matrix L, and stop when it has less than full column rank.

```
> L:= u:
  for nn from 1 do
    L:= <L|Tu[nn]>;
    if LinearAlgebra:-Rank(L) < nn+1 then printf("Success at n=
  %d\n",nn); break fi;
  od:
```

```
Success at n=123
```

The recurrence can then be found from the null space of the matrix L.

```
> P:= LinearAlgebra:-NullSpace(L)[1]:
```

This is the recurrence:

```
> recurrence:= sort(a(n)=solve(add(P[i]*a(n+i-124),i=1..124),a(n)),
  [seq(a(n-i),i=0..124)]);
```

$$
\begin{aligned}
recurrence := a(n) = {} & a(n-3) + 256\, a(n-4) - 234\, a(n-7) - 29565\, a(n-8) + 48\, a(n \\
& -9) + a(n-10) + 24690\, a(n-11) + 2049983\, a(n-12) - 7575\, a(n-13) \\
& - 127\, a(n-14) - 1557996\, a(n-15) - 95670504\, a(n-16) + 529358\, a(n-17) \\
& + 6963\, a(n-18) + 65731976\, a(n-19) + 3189959085\, a(n-20) - 21578736\, a(n \\
& - 21) - 219077\, a(n-22) - 1963100702\, a(n-23) - 78691299725\, a(n-24) \\
& + 568515560\, a(n-25) + 4479400\, a(n-26) + 42856282519\, a(n-27) \\
& + 1467214071523\, a(n-28) - 10118180885\, a(n-29) - 64948762\, a(n-30) \\
& - 696526427396\, a(n-31) - 20953215640751\, a(n-32) + 122968495407\, a(n-33) \\
& + 721379726\, a(n-34) + 8511390899779\, a(n-35) + 231089003119300\, a(n-36) \\
& - 994679515324\, a(n-37) - 6491549315\, a(n-38) - 78581211558184\, a(n-39) \\
& - 1978723617472632\, a(n-40) + 4757097471611\, a(n-41) + 48513957114\, a(n \\
& - 42) + 549662503483287\, a(n-43) + 13206445014484505\, a(n-44) \\
& - 4934815058185\, a(n-45) - 308186400055\, a(n-46) - 2923339623697756\, a(n \\
& - 47) - 68962737088180319\, a(n-48) - 110852650910147\, a(n-49) \\
& + 1717249004346\, a(n-50) + 11899957035553660\, a(n-51) \\
& + 282955491112216298\, a(n-52) + 962748792721999\, a(n-53) \\
& - 8250911744335\, a(n-54) - 37441553449784908\, a(n-55) \\
& - 916803976907447273\, a(n-56) - 4511802819618761\, a(n-57) \\
& + 32202445578770\, a(n-58) + 92037619309192144\, a(n-59) \\
& + 2359156392920749678\, a(n-60) + 14422216753875037\, a(n-61) \\
& - 100132349981353\, a(n-62) - 178366693662249780\, a(n-63) \\
& - 4849803966992106240\, a(n-64) - 33753500133916484\, a(n-65) \\
& + 248993917371312\, a(n-66) + 274107869722579572\, a(n-67) \\
& + 8008739707074120520\, a(n-68) + 59797770362141289\, a(n-69) \\
& - 496080562271724\, a(n-70) - 334629100618654100\, a(n-71) \\
& - 10671065436595352028\, a(n-72) - 81710092798104020\, a(n-73) \\
& + 788818809455360\, a(n-74) + 323634764082995952\, a(n-75) \\
& + 11505534397451674680\, a(n-76) + 87041171425113768\, a(n-77) \\
& - 994435557284196\, a(n-78) - 246033886635763480\, a(n-79) \\
& - 10047350870288632548\, a(n-80) - 72627025559619348\, a(n-81) \\
& + 986028676816864\, a(n-82) + 144978160336522288\, a(n-83) \\
& + 7095794580619393028\, a(n-84) + 47444654378245996\, a(n-85) \\
& - 761327382998124\, a(n-86) - 64754221025744084\, a(n-87) \\
& - 4035402101880564388\, a(n-88) - 24126388872540160\, a(n-89) \\
& + 451550386237396\, a(n-90) + 21158474973657664\, a(n-91) \\
& + 1833778781874894560\, a(n-92) + 9440288199722752\, a(n-93) \\
& - 201842545619744\, a(n-94) - 4761933988708352\, a(n-95)
\end{aligned}
\qquad \mathbf{(3)}
$$

$$
\begin{aligned}
& - 657849885581986816\, a(n-96) - 2790060088588288\, a(n-97) \\
& + 66190674058752\, a(n-98) + 653371441118208\, a(n-99) \\
& + 182994651091525632\, a(n-100) + 605862885359616\, a(n-101) \\
& - 15311522414592\, a(n-102) - 36597596651520\, a(n-103) \\
& - 38446017540259840\, a(n-104) - 92783651061760\, a(n-105) \\
& + 2348458180608\, a(n-106) - 2126433746944\, a(n-107) + 5863693363445760\, a(n \\
& - 108) + 9399720476672\, a(n-109) - 213114683392\, a(n-110) \\
& + 292074553344\, a(n-111) - 609213193650176\, a(n-112) - 561969627136\, a(n \\
& - 113) + 8623489024\, a(n-114) + 38362647887872\, a(n-116) + 15032385536\, a(n \\
& - 117) - 1099511627776\, a(n-120)
\end{aligned}
$$

Note that this is of order 120, but is not true for $n = 121$ **or** 122.