Probability distribution of non-functional points in a random partial functions .

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Definition: Let $m \in [n]$ and $\alpha \in \mathcal{PT}_n$, the semigroup of partial functions on $[n]$. Then m is a *functional point* under α if $m \in M$, where M is the unique maximal subset of $[n]$ such that $\alpha|_M$ is a function. Otherwise m is a *non-functional point* under α.

Let $a_{n,k}$ be the number of partial functions on $[n]$ with exactly k non-functional points.

$$\sum_{n\geq 0} a_{n,k} y^k \frac{x^n}{n!} = \exp(\log(\frac{1}{1-A(x)})) \exp(A(yx))$$

where $A(x)$ is the e.g.f. for the number of rooted labeled trees. By direct counting we have $a_{n,k} = \binom{n}{k}(n-k)^{n-k}(k+1)^{k-1}$. Note that the number of partial functions on $[n]$ is $(n+1)^n$.

Let X_n be the discrete random variable that assigns to each partial function on $[n]$ the number k of its non-functional points, $0 \leq k \leq n$.

$$P(X_n = k) = \frac{\binom{n}{k}(n-k)^{n-k}(k+1)^{k-1}}{(n+1)^n}$$

$$\lim_{n\to\infty} P(X_n = k) = \frac{(k+1)^{k-1}}{e^{k+1}k!}$$

and we have the identity

$$\sum_{k\geq 0} \frac{(k+1)^{k-1}}{e^{k+1}k!} = 1$$

It is perhaps surprising that there is a non-zero limiting distribution. From the distribution, we see that almost all the points in $[n]$ are functional points under a randomly selected $\alpha \in \mathcal{PT}_n$. In particular, no matter how big n gets, the probability that a random partial function has j or fewer non-functional points is

$$P(X \leq j) = \sum_{k=0}^{j} \frac{(k+1)^{k-1}}{e^{k+1} k!}$$

For example, in the case that $j = 10$ the probability is about 76%.