Proof of formula for A184349

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By definition, $a(n)$ is the coefficient of x^{81n} in

$$\prod_{i=1}^{4}\sum_{j=0}^{n} x^{ji^4} = (1+x+\ldots+x^n)(1+x^{16}+\ldots+x^{16n})(1+x^{81}+\ldots+x^{81n})$$
$$= \frac{(1-x^{n+1})}{(1-x)}\frac{(1-x^{16(n+1)})}{(1-x^{16})}\frac{(1-x^{81(n+1)})}{(1-x^{81})}$$

But since all terms in the expansion involving $x^{81(n+1)}$ have exponent greater than $81n$, we can omit the $x^{81(n+1)}$: a_n is the coefficient of x^{81n} in

$$\frac{(1-x^{n+1})(1-x^{16(n+1)})}{(1-x)(1-x^{16})(1-x^{81})} = \frac{1-x^{n+1}-x^{16(n+1)}+x^{17(n+1)}}{(1-x)(1-x^{16})(1-x^{81})}$$

If $b(n)$ is the coefficient of x^n in $1/((1-x)(1-x^{16})(1-x^{81}))$, this can be written as $a(n) = a_1(n) - a_2(n) - a_3(n) + a_4(n)$ where

$$a_1(n) = b(81n)$$
$$a_2(n) = b(80n-1)$$
$$a_3(n) = b(65n-16)$$
$$a_4(n) = b(64n-17)$$

Let $q_i(n) = a_i(n) - 2a_i(n-1) + a_i(n-2) - a_i(n-81) + 2a_i(n-82) + a_i(n-83)$.

The empirical formula $a(n) = 2a(n-1) - a(n-2) - a(n-81) + 2a(n-82) - a(n-83)$ corresponds to $q_1(n) - q_2(n) - q_3(n) + q_4(n) = 0$ for $n \geq 84$. In fact, I will show that $q_1(n) = q_3(n)$ while $q_2(n) = q_4(n) = 0$.

If T is the shift operator $Tb(n) = b(n+1)$, we have $(T-1)(T^{16}-1)(T^{81}-1)b = 0$

Thus for $n \geq 84$,

$$q_1(n) = (T^{81n} - 2T^{81n-81} + T^{81n-162} - T^{81n-6561} + 2T^{81n-6642} - T^{81n-6723})(b)(0)$$
$$= T^{81n-6723}(T^{81}-1)^2(T^{81^2}-1)(b)(0)$$

and

$$q_1(n+16) - q_1(n) = (T^{81\cdot 16} - 1)T^{81n-6723}(T^{81}-1)^2(T^{81^2}-1)(b)(0)$$

Noting that $(z^{81\cdot 16}-1)z^{81n-6723}(z^{81}-1)^2(z^{81^2}-1)$ is divisible (as a polynomial in z) by $(z^{81}-1)(z^{16}-1)(z-1)$, we see that $q_1(n+16)-q_1(n)=0$, i.e. q_1 is periodic with period 16.

Similarly,

$$q_3(n) = T^{65n-5411}(T^{65}-1)^2(T^{65\cdot 81}-1)(b)(0)$$

and $(z^{65*16}-1)z^{65n-5411}(z^{65}-1)^2(z^{65\cdot 81}-1)$ is divisible by $(z^{81}-1)(z^{16}-1)(z-1)$, so q_3 is periodic with period 16. Moreover, it turns out (by computation) that

$$q_1(n) = q_3(n) = (0,0,0,0,0,0,0,0,0,0,0,0,1,-1,0,0) \text{ for } n = 84\ldots 84+15$$

and so $q_1(n) = q_3(n)$ for all $n \geq 84$.

Next,

$$q_2(n) = T^{80n-6641}(T^{80}-1)^2(T^{80\cdot 81}-1)(b)(0)$$

with $z^{80n-6641}(z^{80}-1)^2(z^{80\cdot 81}-1)$ divisible by $(z^{81}-1)(z^{16}-1)(z-1)$, so $q_2(n)=0$. Similarly,

$$q_4(n) = T^{64n-5329}(T^{64}-1)^2(T^{64\cdot 81}-1)(b)(0)$$

with $z^{64n-5329}(z^{64}-1)^2(z^{64\cdot 81}-1)$ divisible by $(z^{81}-1)(z^{16}-1)(z-1)$, so $q_4(n)=0$.

This completes the proof.