Proof of emprirical formula for A183978

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The 2×2 subblock consisting of rows i and $i+1$ has the same sum as the subblock consisting of rows $i+1$ and $i+2$ if and only if rows i and $i+2$ have the same sum. Thus an equivalent condition is that all odd-numbered rows have the same sum and all even-numbered rows have the same sum. Those row sums can only be 0, 1 or 2, with row sum 0 equiring a row of $(0,0)$, row sum 2 requiring $(1,1)$, and 1 allowing either $(0,1)$ or $(1,0)$. Thus considering all possibilities for $\lfloor (n+1)/2 \rfloor$ even-numbered and $\lceil (n+1)/2 \rceil$ odd-numbered rows, we get

$$
\begin{aligned}
A183978(n) &= (2 + 2^{\lfloor (n+1)/2 \rfloor})(2 + 2^{\lceil (n+1)/2 \rceil})/4 \\
&= (1 + 2^{\lfloor (n-1)/2 \rfloor})(1 + 2^{\lceil (n-1)/2 \rceil}) \\
&= \begin{cases} 2^{2k} + 2^{k+1} + 1 & \text{if } n = 2k+1 \text{ is odd} \\ 2^{2k-1} + 3 \cdot 2^{k-1} + 1 & \text{if } n = 2k \text{ is even} \end{cases}
\end{aligned}
$$

It is then straightforward to verify the recurrence and generating function.