Columns of A183912 are eventually polynomials

Robert Israel

April 5, 2018

Theorem 1 *Each column of sequence A183912 is eventually equal to a polynomial in n.*

Proof A column of A183912 can be considered as the enumeration of certain walks on a directed graph (with loops allowed). The nodes of the graph are labelled by nondecreasing strings in $0 \dots k$ with each number occurring at most 3 times. I'll write i^2 and i^3 for ii and iii. From the empty sequence $\emptyset$ we have arcs to j for all $j \in \{0, \dots, k\}$. From a sequence S ending in i^s we have arcs to Sj for each $j > i$ and to Si if $s < 3$ or a loop $S \to S$ if $s = 3$.

The "good" nodes are those S such that if S contains i, it must contain i' and i'' with $i \equiv i' + i'' \bmod (k+1)$, with $i' = i'' = i'''$ allowed only if S contains i^3, $i = i' \neq i''$ allowed only if it contains i^2, and $i' = i''$ allowed only if it contains i'^2. This is because in the definition of A183912, each number must be the sum mod $k+1$ of "two others", which is interpreted as meaning two members of the string, distinct from each other and the first member (although possibly with the same numerical value).

The adjacency matrix T of this directed graph is upper triangular (for a suitable ordering of the nodes), with diagonal elements 0 and 1, so it has eigenvalues 0 and 1 and its minimal polynomial is of the form $x^a(x-1)^b$ for some nonnegative integers a and b. Then $a_n = v^T T^{n+2} u$ where v is the vector with $v_\emptyset = 1$ and all other $v_S = 0$, and u the vector with $u_S = 1$ if S is "good" and 0 otherwise. Thus the b'th difference of a_n is 0 for $n \geq a - 2$, implying that a_n is equal to a polynomial of degree $\leq b - 1$ for $n \geq a - 2$.

Example For $k = 1$, the graph has 16 nodes:

$$\emptyset, 0, 0^2, 0^3, 1, 1^2, 1^3, 01, 0^21, 0^31, 01^2, 0^21^2, 0^31^2, 01^3, 0^21^3, 0^31^3$$

The good nodes are $\emptyset, 0^3, 01^2, 0^21^2, 0^31^2, 01^3, 0^21^3, 0^31^3$. The matrix

$$T = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

has minimal polynomial $x^5(1-x)^2$. Thus a_n is equal to a polynomial of degree ≤ 2 for $n \geq 3$. However, it turns out that $v^T T^n u = n - 1$ for $n \geq 3$, i.e. $a_n = n + 1$ for $n \geq 1$.