Maple-assisted derivation of recurrence for A183520

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There are $3^6 = 729$ possible configurations for a 2×3 array. Consider the 729×729 transition matrix T such that $T_{ij} = 1$ if the top two rows of a 3×6 sub-array could be in configuration i while the bottom two rows are in configuration j (so the middle row is common to both, and each element there is the sum mod 3 of its horizontal and vertical or its diagonal and antidiagonal neighbours, and 0 otherwise. The following Maple code computes it. I'm encoding a 2×3 array as $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$

```
> Configs:= map(t -> convert(t+3^6, base, 3)[1..6],[$0..3^6-1]):
```

```
> q:= proc(a,b) local A,B;
     A:= Configs[a]; B:= Configs[b];
     if A[4..6]<> B[1..3] then return 0 fi;
     if A[4] <> A[1]+A[5]+B[4] mod 3  and A[4] <> A[2]+B[5] mod 3
  then return 0 fi;
     if A[5] <> A[2]+A[4]+A[6]+B[5] mod 3 and A[5] <> A[1]+A[3]+B
  [4]+B[6] mod 3 then return 0 fi;
     if A[6] <> A[3]+A[5]+B[6] mod 3 and A[6] <> A[2]+B[5] mod 3
  then return 0 fi;
     1
  end proc:
  T:= Matrix(3^6,3^6, q):
```

Thus for $n \geq 1$, $a(n) = u^T T^n v$ where u and v are column vectors with entries 1 for configurations where the top row and the bottom row is 0, respectively, and otherwise 0.

```
> uT:= Vector[row](3^6,proc(a) if Configs[a][1..3]=[0,0,0] then 1
  else 0 fi end proc):
  v:= Vector(3^6,proc(a) if Configs[a][4..6]=[0,0,0] then 1 else 0
  fi end proc):
```

To check, here are the first few entries of our sequence. For future use, we pre-compute $T^n v$.

```
> V[0]:= v:
  for nn from 1 to 200 do V[nn]:= T . V[nn-1] od:
> seq(uT. V[n], n = 1 .. 30);
```

5, 31, 101, 543, 2233, 10003, 47685, 215451, 994397, 4603823, 21240401, 98257363, 454235165, 2100740935, 9717553917, 44943606231, 207898873245, 961691911899, 4448501263357, 20578124472715, 95191234404373, 440340646073843, 2036961372499565, 9422737080433915, 43588479047668973, 201635356321866775, 932742687142041629, 4314764828063477251, 19959629263126736233, 92331071936304234687 **(1)**

To find a linear recurrence, we see when the vectors v, Tv, ..., $T^n v$ are linearly dependent.

```
> M:= v:
  for nn from 1 to 200 do
    M:= <M|V[nn]>;
```

```
   if LinearAlgebra:-Rank(M) < nn+1 then printf("Success for n=
  %d\n",nn); break fi;
  od:
Success for n=157
```

The recurrence is then obtained from the null space of the matrix formed by these vectors.

```
> W:= LinearAlgebra:-NullSpace(M)[1]:
```

It turns out that $W_1 = 0$, so the recurrence is of order 156.

```
> W[1];
```

$$0 \tag{2}$$

```
> Rec:= sort(add(W[i]*a(k+i-2),i=1..158),[seq(a(k+i-2),i=1..158)])=
  0;
```

$$Rec := -25613880561021606690816 00\,a(k) - 770490033759354723552460 80\,a(k+1) \tag{3}$$
$$- 3160183337931629814727311 36\,a(k+2) + 1100480855039041142544924672\,a(k+3) + 2595445099836205573550899200\,a(k+4)$$
$$- 3367872150369771228932079616\,a(k+5) - 7877288559719682725610782720\,a(k+6) - 2925556734294970797468418048\,a(k+7)$$
$$+ 12207852108388671916004081664\,a(k+8) + 19377049870889473810819448832\,a(k+9) + 19428547628461295834278133 76\,a(k+10)$$
$$- 19778686674848397489364008960\,a(k+11)$$
$$- 28037538742483992984768479232\,a(k+12) + 21401196675489715649277296 64\,a(k+13) + 16941545739517085077620867072\,a(k+14)$$
$$+ 10420976691219718882014167040\,a(k+15) - 39394656555390990654779392 00\,a(k+16) - 10179731335004709015698964 48\,a(k+17)$$
$$+ 4357554009108113217531910144\,a(k+18) - 5564240293105458990098998272\,a(k+19) + 10869319584082331204074271744\,a(k+20)$$
$$+ 27001378109369146778260411904\,a(k+21) - 94065703968533189107912965 12\,a(k+22) - 30520887840521442220642322688\,a(k+23)$$
$$- 10960900361409092995263526144\,a(k+24) - 75121509363204356404889437 44\,a(k+25) - 7734239186438484371509422 08\,a(k+26)$$
$$+ 17955620266802583423690168064\,a(k+27)$$
$$+ 10315716642150103484994694592\,a(k+28) - 45392830812950424325973194 88\,a(k+29) + 14444748467122401098497307 52\,a(k+30)$$
$$- 1481301068183270495815033808\,a(k+31) - 3482004251140012573557448432\,a(k+32) + 3712762077683286679683838800\,a(k+33)$$
$$+ 2362104346657670768890599008\,a(k+34) - 1840997231825759895382499160\,a(k+35) + 635281536436489060303527556\,a(k+36)$$
$$- 534728345565988594854190144\,a(k+37) - 191831811519572094387703276 8\,a(k+38) + 270281369435284120959828512\,a(k+39)$$
$$- 134239949870593185831649750\,a(k+40) - 62266647395507375043574818 0\,a(k+41) + 184692298808030792293602866\,a(k+42)$$
$$+ 38128371330182037428826162 6\,a(k+43) + 11277647728823460989468563\,a(k+44) - 71086173949037692638130033\,a(k+45)$$
$$+ 13312623512155224243335734 7\,a(k+46) - 16555785704413713557037966\,a(k+47) + 10402730008974919451433595 9\,a(k+48)$$

$+ 6093512329506760901955492 0\, a(k+49) - 4548003569276730916202093 3\, a(k$

$+ 50) + 8883588084527383476449 63\, a(k+51) + 1312848810373514436554633 5\, a(k$

$+ 52) - 1092039022757568078299424 0\, a(k+53)$

$- 1555721393805347124297966 5\, a(k+54) + 1637392232900581052681632\, a(k+55)$

$- 2193760861361967674972155\, a(k+56) - 4665182295192152093862572\, a(k+57)$

$- 1783081530634384555850606\, a(k+58) - 1481674458667282187830430\, a(k+59)$

$+ 677029505422844291129492\, a(k+60) + 641429095480021676336013\, a(k+61)$

$- 101688348088497652710871\, a(k+62) + 133162677342296914 22579\, a(k+63)$

$+ 843747295673404651515474\, a(k+64) + 272909793877748599433803\, a(k+65)$

$- 167018547594340259562327\, a(k+66) + 261591796462178794907 41\, a(k+67)$

$+ 87832345790927860016811\, a(k+68) - 55709913411021392694079\, a(k+69)$

$- 38249319915267454715182\, a(k+70) - 11011353679143579495625\, a(k+71)$

$- 10314155799565902970608\, a(k+72) - 3375661247588051281611\, a(k+73)$

$- 4650865271464657858637\, a(k+74) + 1179119683501878969154\, a(k+75)$

$+ 3495562379154822826353\, a(k+76) + 2124846101902787602727\, a(k+77)$

$+ 920876593085151985931\, a(k+78) + 1045549253675034741085\, a(k+79)$

$+ 670728319008985813374\, a(k+80) - 248611538453314872897\, a(k+81)$

$- 114089976601368798655\, a(k+82) - 74198473570294681649\, a(k+83)$

$- 154911067614781003085\, a(k+84) - 18180737067663436 2316\, a(k+85)$

$+ 13401796752480348249\, a(k+86) - 13085957902324091208\, a(k+87)$

$- 31555379176785648557\, a(k+88) + 3271214117831083322\, a(k+89)$

$+ 2181369404716428901 6\, a(k+90) + 5758441714980051803\, a(k+91)$

$- 2709106808626704230\, a(k+92) + 4301867205300990375\, a(k+93)$

$+ 3849683713131349387\, a(k+94) - 920194474879762536\, a(k+95)$

$- 1114308768458078085\, a(k+96) + 596847364531653681\, a(k+97)$

$- 115582139483861326\, a(k+98) - 418199401917028307\, a(k+99)$

$- 115144828531162410\, a(k+100) + 51316729574978283\, a(k+101)$

$- 30741678627481593\, a(k+102) - 32340440679818967\, a(k+103)$

$+ 11959867393390996\, a(k+104) + 12245986656136582\, a(k+105)$

$- 139186832912364\, a(k+106) + 1067681197661017\, a(k+107)$

$+ 2436634179334821\, a(k+108) + 1055384646434213\, a(k+109)$

$- 475431278971\, a(k+110) - 14587495544962\, a(k+111) + 113331162673890\, a(k$

$+ 112) - 24997568028242\, a(k+113) - 55917581616454\, a(k+114)$

$- 21238723541765\, a(k+115) - 13720588742392\, a(k+116) - 9670492815688\, a(k$

$+ 117) - 5495411715627\, a(k+118) - 2041898533664\, a(k+119)$

$- 577062754545\, a(k+120) + 190782479752\, a(k+121) + 251111888225\, a(k+122)$

$+ 192352834495\, a(k+123) + 148062637670\, a(k+124) + 110504245767\, a(k+125)$

$+ 40422745910\, a(k+126) + 12651556342\, a(k+127) + 3179193845\, a(k+128)$

$- 660032865\, a(k+129) - 1812756206\, a(k+130) - 1961187858\, a(k+131)$

$- 1009861370\, a(k+132) - 327656849\, a(k+133) - 156454544\, a(k+134)$

$- 51533822\, a(k+135) + 18276217\, a(k+136) + 13148091\, a(k+137)$

$+ 10969601\, a(k+138) + 5723970\, a(k+139) + 2029861\, a(k+140) + 499469\, a(k$

$$+ 141) + 128412\, a(k + 142) - 92232\, a(k + 143) - 64812\, a(k + 144) - 51483\, a(k + 145) - 14784\, a(k + 146) - 3051\, a(k + 147) + 115\, a(k + 148) - 83\, a(k + 149) + 504\, a(k + 150) + 278\, a(k + 151) + 64\, a(k + 152) - 24\, a(k + 153) - 9\, a(k + 154) - 3\, a(k + 155) + a(k + 156) = 0$$