Maple-assisted derivation of recurrence for A183334

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There are $2^{12}=4096$ possible configurations for two adjacent rows, but only those where each 1 is adjacent to at most two 0's is allowed. The following code computes the allowed configurations, of which there turn out to be 2584.

```
> Configs:= NULL: nConfigs:= 0:
  for n from 0 to 4095 do
    v:= convert(4096+n,base,2)[1..12];
    good:= true:
    for i from 1 to 12 do
      if v[i] = 1 then
        t:= 0:
        if i mod 6 <> 1 then t:= t + (1-v[i-1]) fi;
        if i mod 6 <> 0 then t:= t + (1-v[i+1]) fi;
        if i <= 6 then t:= t + 1-v[i+6] else t:= t + 1-v[i-6] fi;
        if t > 2 then good:= false; break fi
      fi
    od;
    if good then Configs:= Configs, v; nConfigs:= nConfigs+1; fi
  od:
  Configs:= [Configs]:
  nConfigs;
```

2584 **(1)**

Let T be the 2584×2584 transition matrix such that $T_{ij}=1$ if the bottom two rows of a 3×6 array could be in configuration i while the top two rows are in configuration j (thus the middle row is both the top row of configuration i and the bottom row of configuration j, and every 1 in the middle row has exactly two neighbours that are 0's). The following code computes it.

```
> q:= proc(i,j) local Ci, Cj,k;
     Ci:= Configs[i]; Cj:= Configs[j];
     if Ci[1..6] <> Cj[7..12] then return 0 fi;
     if Ci[1] = 1 and Ci[7]+Ci[2]+Cj[1] <> 1 then return 0 fi;
     if Ci[6] = 1 and Ci[5]+Ci[12]+Cj[6] <> 1 then return 0 fi;
     for k from 2 to 5 do
       if Ci[k] = 1 and Ci[k-1]+Ci[k+1]+Ci[k+6]+Cj[k] <> 2 then
  return 0 fi
     od;
     1
  end proc:
  T:= Matrix(nConfigs,nConfigs, q):
```

Let u be the column vector with 1's corresponding to configurations that could be the first two rows of an array, in that each 1 in the first row is adjacent to exactly two 0's, 0 otherwise, and similarly v the column vector with 1's corresponding to configurations that could be the bottom two rows.

Then $a(n)=v^T T^{n-2} u$ for $n \geq 2$.

```
> u:= Vector(nConfigs, proc(i) local Ci,k;
    Ci:= Configs[i];
    if Ci[1] = 1 and Ci[2]+Ci[7] <> 0 then return 0 fi;
```

```
    if Ci[6] = 1 and Ci[5]+Ci[12] <> 0 then return 0 fi:
    for k from 2 to 5 do if Ci[k] = 1 and Ci[k-1]+Ci[k+1]+Ci[k+6]
  <> 1 then return 0 fi od:
    1 end proc):
  v:=  Vector(nConfigs, proc(i) local Ci,k;
    Ci:= Configs[i];
    if Ci[7] = 1 and Ci[1]+Ci[8] <> 0 then return 0 fi;
    if Ci[12] = 1 and Ci[11]+Ci[6] <> 0 then return 0 fi:
    for k from 8 to 11 do if Ci[k] = 1 and Ci[k-1]+Ci[k+1]+Ci[k-6]
  <> 1 then return 0 fi od:
    1 end proc):
```

To check, here are the first few entries of our sequence. For future use, I compute $U_n = T^n\, u$ for some values of n.

```
> U[0]:= u:
  for n from 1 to 100 do U[n]:= T . U[n-1] od:
> seq(v^%T . U[n], n = 0 .. 18);
```

98, 534, 3254, 21708, 135578, 863096, 5518572, 35095832, 223543018, 1424141918, 9069450042, 57765518688, 367927491536, 2343383959002, 14925484493108, 95063475440328, 605477794684548, 3856408602298962, 24562234175026876 **(2)**

To find a recurrence, we check when the vectors U_0, ..., U_r fail to be linearly independent.

```
> M:= U[0]:
  for r from 1 do
    M:= <M|U[r]>;
    if LinearAlgebra:-Rank(M) < r+1 then printf("dependent for r=
  %d\n",r); break fi
  od:
dependent for r=98
```

The recurrence corresponds to a vector in the null space of the matrix formed by those columns.

```
> R:= op(1,LinearAlgebra:-NullSpace(M)):
> convert(R,list);
```

[2, 6, −19, 56, 51, 73, 325, −745, 3161, −1087, 7409, 3810, 14364, 42340, −49096, 107767, −272615, 300546, −933960, −39917, −1654115, −923241, −816698, −2539666, 1363365, −4801852, 1723472, −6411977, 2391569, 10550230, −9211606, 8727661, −10762457, −17645727, 51464597, −17474154, 41542508, 36622443, −63666568, 26850055, 34169760, 25380599, 37001995, −84219398, −137403682, −22113297, 25606819, 30978919, 212721225, 24330864, −240026344, −57805190, −45388428, 14477172, 111699571, −51743322, 87642736, 155708375, −104651527, −172789133, −52181415, 137306554, 122004802, −140041878, −157342254, 99520013, 220082699, 60384138, −140370120, −129543752, 3971631, 56195341, 26652336, −1550713, −9173361, −4799589, 2257270, 3131050, −522464, −1751202, 104943, 914503, −3929, −432564, −99093, 87297, 36149, −11713, −3317, 6248, 1589, 99, −225, −201, 15, −46, −1, −5, 1] **(3)**

If this vector is $[c_0, c_1, ..., c_r]$, it means that $\sum_{i=0}^{r} c_i\, T^i u = 0$ so that $\sum_{i=0}^{r} c_i\, a(n+i) = \sum_{i=0}^{r} c_i\, v^T\, T^{n+i-2}\, u = 0$ for $n \geq 2$. Here is the recurrence:

```
> n:= 'n':
```

```
a(n) = sort(- add(R[i+1]/R[r+1] * a(n+i-r),i=0..r-1),[seq(a(n-i),
i=1..r)]);
```

$$a(n) = 5\,a(n-1) + a(n-2) + 46\,a(n-3) - 15\,a(n-4) + 201\,a(n-5) + 225\,a(n-6) - 99\,a(n-7) - 1589\,a(n-8) - 6248\,a(n-9) + 3317\,a(n-10) + 11713\,a(n-11) - 36149\,a(n-12) - 87297\,a(n-13) + 99093\,a(n-14) + 432564\,a(n-15) + 3929\,a(n-16) - 914503\,a(n-17) - 104943\,a(n-18) + 1751202\,a(n-19) + 522464\,a(n-20) - 3131050\,a(n-21) - 2257270\,a(n-22) + 4799589\,a(n-23) + 9173361\,a(n-24) + 1550713\,a(n-25) - 26652336\,a(n-26) - 56195341\,a(n-27) - 3971631\,a(n-28) + 129543752\,a(n-29) + 140370120\,a(n-30) - 60384138\,a(n-31) - 220082699\,a(n-32) - 99520013\,a(n-33) + 157342254\,a(n-34) + 140041878\,a(n-35) - 122004802\,a(n-36) - 137306554\,a(n-37) + 52181415\,a(n-38) + 172789133\,a(n-39) + 104651527\,a(n-40) - 155708375\,a(n-41) - 87642736\,a(n-42) + 51743322\,a(n-43) - 111699571\,a(n-44) - 14477172\,a(n-45) + 45388428\,a(n-46) + 57805190\,a(n-47) + 240026344\,a(n-48) - 24330864\,a(n-49) - 212721225\,a(n-50) - 30978919\,a(n-51) - 25606819\,a(n-52) + 22113297\,a(n-53) + 137403682\,a(n-54) + 84219398\,a(n-55) - 37001995\,a(n-56) - 25380599\,a(n-57) - 34169760\,a(n-58) - 26850055\,a(n-59) + 63666568\,a(n-60) - 36622443\,a(n-61) - 41542508\,a(n-62) + 17474154\,a(n-63) - 51464597\,a(n-64) + 17645727\,a(n-65) + 10762457\,a(n-66) - 8727661\,a(n-67) + 9211606\,a(n-68) - 10550230\,a(n-69) - 2391569\,a(n-70) + 6411977\,a(n-71) - 1723472\,a(n-72) + 4801852\,a(n-73) - 1363365\,a(n-74) + 2539666\,a(n-75) + 816698\,a(n-76) + 923241\,a(n-77) + 1654115\,a(n-78) + 39917\,a(n-79) + 933960\,a(n-80) - 300546\,a(n-81) + 272615\,a(n-82) - 107767\,a(n-83) + 49096\,a(n-84) - 42340\,a(n-85) - 14364\,a(n-86) - 3810\,a(n-87) - 7409\,a(n-88) + 1087\,a(n-89) - 3161\,a(n-90) + 745\,a(n-91) - 325\,a(n-92) - 73\,a(n-93) - 51\,a(n-94) - 56\,a(n-95) + 19\,a(n-96) - 6\,a(n-97) - 2\,a(n-98) \quad \textbf{(4)}$$

We have proven it for $n \geq 100$, but it turns out to also be true for $n = 99$.